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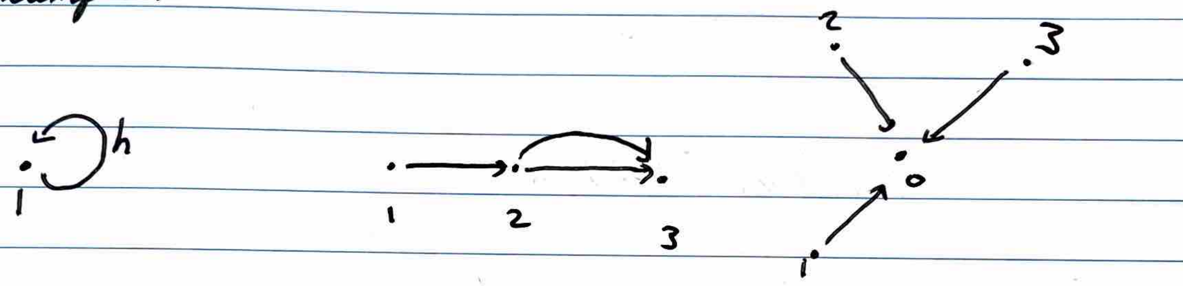
Quivers : Lecture 1

Definition A quiver is a directed multigraph.
This means it is a 2-tuple (I, Ω) where
- I is the set of vertices.
- Ω is the set of arrows.

A vertex is just a label : x, 1, 0, etc.
An arrow is just a pair of vertices : (x,y), $(1,2)$, etc.
we often write : $x \to y$, $1 \to 2$, ...

For example :



are all examples of quivers.

Assumptions (1) I and Ω are finite
(2) $\vec{Q}$ is connected
(3) $\underline{k}$ is a field.

Definition A representation $V = (\{V_i\}_{i \in I}, \{x_h\}_{h \in \Omega})$ of a quiver $\vec{Q}$ is
- A k-vector space V_i for each vertex $i \in I$.
- A linear map $x_h : V_i \to V_j$ for each arrow $h : i \to j \in \Omega$.

For example :

A representation of $\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$ is $\mathbb{R}^2 \xrightarrow{f} \mathbb{R}^3$

where $f(x,y) = (x, y, 0)$.

Definition Let $V = (\{V_i\}, \{x_h\})$ and $W = (\{W_i\}, \{y_h\})$ be representations of a quiver $\vec{Q}$. A morphism of quiver representations from V to W, $\{f_i\}_{i \in I}$, is
- A linear map $f_i : V_i \to W_i$ for each vertex $i \in I$

such that for each arrow $h : i \to j \in \Omega$, the square

$$\begin{tikzcd} V_i \arrow[r, "x_h"] \arrow[d, "f_i"'] & V_j \arrow[d, "f_j"] \\ W_i \arrow[r, "y_h"'] & W_j \end{tikzcd}$$

commutes.

This means that $f_j \cdot x_h = y_h f_i$.

We will see examples of this in a moment.

Definition A morphism of quiver representations $f : V \to W$ is an isomorphism if each of the linear maps f_i are bijective.

Assumption (1) All quiver representations are finite-dimensional, meaning that all the underlying vector spaces are finite-dimensional.

Example (Jordan Quiver)

Define $\vec{Q} = \begin{tikzcd} 1 \arrow[loop right, "h"] \end{tikzcd}$

A representation of $\vec{Q}$ is thus a pair consisting of a vector space V_1 and a linear map $x : V_1 \to V_1$.

A natural question to ask is: What are the isomorphism classes of representations of $\vec{Q}$?

To examine this, we perhaps would like to consider what happens when we have two isomorphic quiver representations.

Suppose that $V=(V_1, x)$ and $W=(W_1, y)$ are isomorphic representations of $\vec{Q}$. Then, there exists a bijective linear map $f: V_1 \to W_1$ such that the square

$$\begin{tikzcd} V \arrow[r, "x"] \arrow[d, "f"'] & V \arrow[d, "f"] \\ W \arrow[r, "y"'] & W \end{tikzcd}$$

commutes.

This means that $fx = yf$.
Since f is bijective, hence invertible, we conclude that x and y are related by the equation

$$y = fxf^{-1}.$$

So, isomorphic representations of the Jordan quiver are representations that:
- Share the same vector space (up to isomorphism)
- Share the same linear map up to a change of basis.

Thus, classifying representations of the Jordan quiver up to isomorphism is the same problem as classifying linear maps up to a change of basis. This problem (over an algebraically closed field) is solved by the Jordan Normal Form.

Example (A_2 Quiver)

Define $\vec{Q} = \underset{1}{\cdot} \xrightarrow{h} \underset{2}{\cdot}$.

A representation of $\vec{Q}$ is thus a 3-tuple consisting of vector spaces V_1, V_2 and a linear map $x: V_1 \to V_2$.

Suppose that $V = (V_1, V_2, x)$ and $W = (W_1, W_2, y)$ are isomorphic representations of $\vec{Q}$. Then, there exists two bijective linear maps $f_1: V_1 \to W_1$ and $f_2: V_2 \to W_2$ such that the square

$$\begin{tikzcd} V_1 \arrow[r, "x"] \arrow[d, "f_1"'] & V_2 \arrow[d, "f_2"] \\ W_1 \arrow[r, "y"'] & W_2 \end{tikzcd}$$

commutes.

This means that $f_2 x = y f_1$.
Since f_1 is bijective, hence invertible, we conclude that x and y are related by the equation

$$y = f_2 x f_1^{-1}.$$

So, isomorphic representations of the A_2 quiver are representations that:

- Share the same vector spaces (up to isomorphism)
- Have linear maps x, y that are related such that

$$y \in GL(\dim(V_2))\, x\, GL(\dim(V_1)).$$

If $\mathrm{rank}(x) = r$, then a particularly nice representative of the set $GL(\dim(V_2)) \times GL(\dim(V_1))$, when written as a matrix with respect to some basis, is

$$\underbrace{\begin{pmatrix} I_{r\times r} & 0 \\ 0 & 0 \end{pmatrix}}_{\dim(V_1)} \Big\} \dim(V_2) .$$

Thus, isomorphism classes of representations of the A_2 quiver with $\dim(V_1)$ and $\dim(V_2)$ fixed are in bijection with integers r which satisfy $0 \leq r \leq \min(\dim(V_1), \dim(V_2))$.