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22^{nd} Sep 2025

From Dimers to Webs
Meeting 3

Speaker: Jakob Sloan

Let $n \in \mathbb{Z}_{\geq 0}$

$$\begin{aligned} P : Gr(k;n) &\hookrightarrow \mathbb{P}^{\binom{n}{k}-1} \\ V &\longmapsto [P_I(X_V)]_{I \in \binom{[n]}{k}} \end{aligned}$$

where X_V is a matrix with rowspan equal to V.

Recall $X_V \xrightarrow{RREF} \left(\underbrace{\begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix}}_{k\times k \text{ block}} \middle| \underbrace{\begin{matrix} A \end{matrix}}_{k\times n-k \text{ block}} \right)$

The Plücker embedding is about comparing bases.
More than that, it's about comparing subbases (lin. indep. subsets)

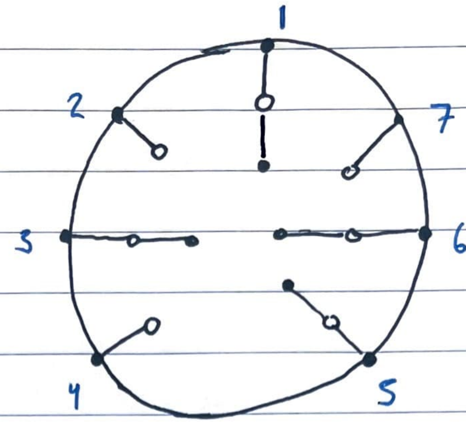
The object that deals with such ideas are called matroids. (algebraic)
We would like some diagrammatic calculus (dimers).

The map of ideas is:

$$\begin{tikzcd}
\{\text{positroids}\} \arrow[r, squiggly] \arrow[d, phantom, "\cap"] \arrow[dr, squiggly, "\text{geometric (normal and Cohen-Mac)}"] & Gr(k;n)_{\geq 0} & \\
\{\text{matroids}\} \arrow[d, phantom, "\text{mysterious}"] & Gr(k;n) \arrow[l, squiggly] \arrow[r, hook, "P"] \arrow[dr] & \mathbb{P}^{\binom{n}{k}-1} \arrow[d, phantom, "\cup"] \\
\{\text{dimer configs}\} \arrow[ur, dotted] \arrow[rr, "(\Delta_I)_{I \in \binom{[n]}{k}}"'] & & \operatorname{im}(P)
\end{tikzcd}$$

Example (Lollipops) Let $n=7$ and $k=3$.

Define $N^{(2,4,7)}_{PAI} =$

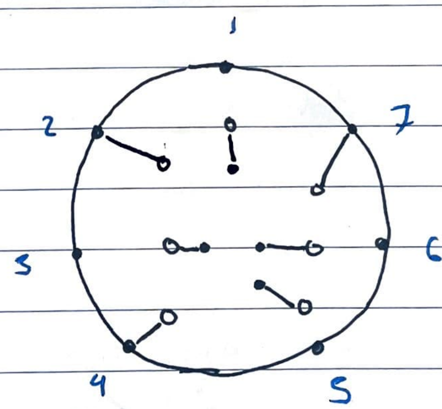


$N^{(2,4,7)}_{PAI}$ has one dimer configuration π.
Such a configuration is obtained by removing edges so that:

- each interior vertex is connected to exactly one vertex
- each boundary vertex is connected to at most one vertex.

This unique configuration is

$\pi^{(2,4,7)} =$

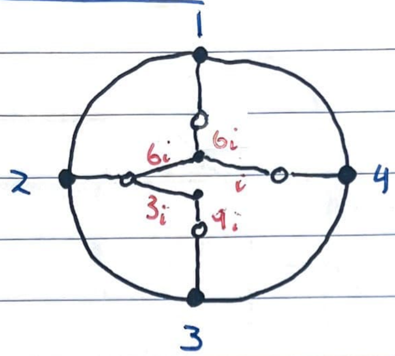


Here, $\Delta_I(N^{(2,4,7)}_{PAI}) = \begin{cases} 1, & I = 247 \\ 0, & \text{otherwise} \end{cases}$.

and $N^{(2,4,7)}_{PAI}$ corresponds to the matrix $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$

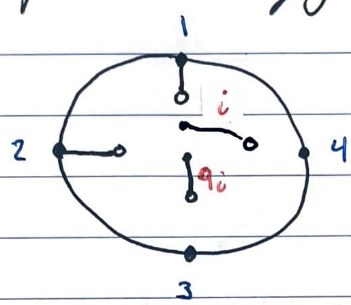
Example (Favourite) Let $n=4$ and $k=2$.

$N_{fav} =$

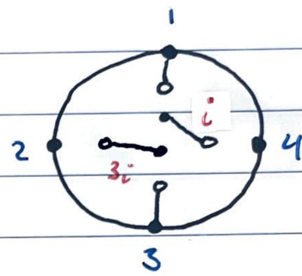


N_{fav} has five dimer configurations

$\pi^{(12)} =$

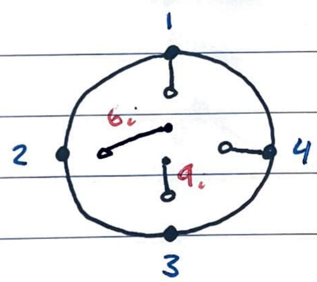


$\pi^{(13)} =$

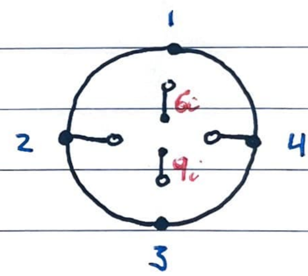


$wt(\pi^{(12)}) = (i)(9i) = -9$ $\quad$ $wt(\pi^{(13)}) = (i)(3i) = -3$

$\pi^{(14)} =$

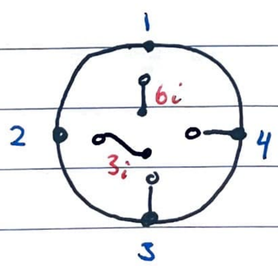


$\pi^{(24)} =$



$wt(\pi^{(14)}) = (6i)(9i) = -54$ $\quad$ $wt(\pi^{(24)}) = (6i)(9i) = -54$

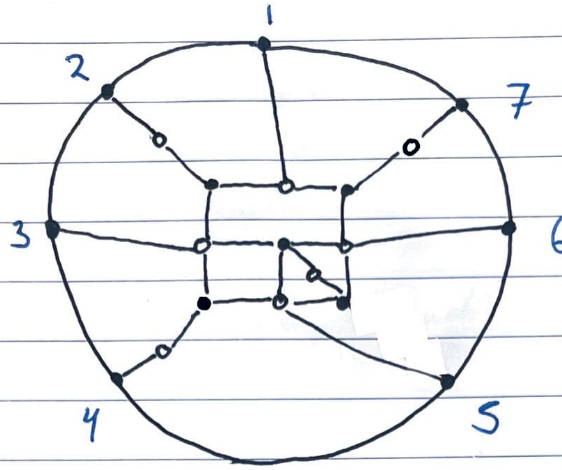
$\pi^{(34)} =$



$wt(\pi^{(34)}) = (6i)(3i) = -18$

Example Let $n=7$ and $k=3$

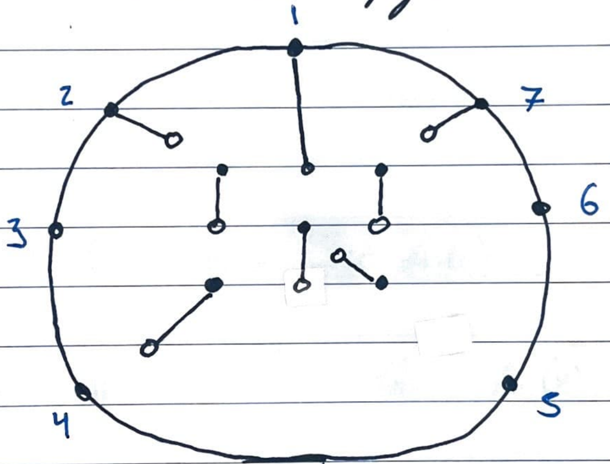
$N =$



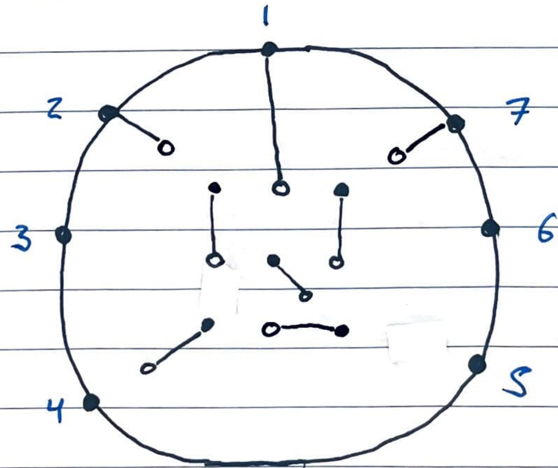
(maybe more?)

N has two $(1,2,7)$ dimer configurations:

$\pi^{(127)}_{(1)} =$



$\pi^{(127)}_{(2)}$



A dimer configuration is a triple

$$(N, \mathrm{wt}_E, \pi) \quad (\text{often just written } \pi)$$

such that

(I) $N=(V,E)$ is a finite graph such that

$V = V_\bullet \sqcup V_\circ$

and if $e=(v,w)\in E$, then (i) $v\in V_\bullet$ and $w\in V_\circ$, or
(ii) $v \in V_\circ$ and $w \in V_\bullet$ — bipartite

equipped with

$j : N \hookrightarrow \overline{\mathbb{D}}$ (closed unit disk)

such that if $j(v)\in\partial(\overline{\mathbb{D}})$, then $\deg(v)=1$.

Additionally, $\partial(V) = \{ j \in V \mid j(v) \in \partial(\overline{\mathbb{D}}) \} \subseteq V_\bullet$

boundary vertices are black with degree 1.

(II) $\mathrm{wt} : E \to \mathbb{C}^\times$
(convention is that if $\mathrm{wt}(e)=1$ then you don't write it down.
c.f boundary edges on pg. 3)

(III) $\pi \subseteq E$ satisfies

if $v \in V^\circ = \mathrm{im}(j)\setminus\partial(V)$ (interior), then there exists unique $w\in V$ such that $(v,w)\in\pi$.

Remarks

(1) (N, wt_E) is called a network.

(2) A boundary vertex is an element of $\partial(V)$.
A boundary edge is an edge $(v,w) \in E$ with $v \in \partial(V)$ or $w \in \partial(V)$.

(3) If N is a network and π_1, π_2 are dimers on N, then

$$\mathrm{Card}(\partial(\pi_1)) = \mathrm{Card}(\partial(\pi_2))$$

where $\partial(\pi) = \pi \cap \{\text{boundary edges}\}$ (i.e. # boundary edges in π)

This number is called the excedance of N.

(4) The following statement holds:

$$\mathrm{Card}(\partial(\pi)) = \mathrm{Card}(\mathring{V} \cap V_\bullet) - \mathrm{Card}(\mathring{V} \cap V_\circ)$$

(5) Let N be a network with $\mathrm{Card}(\partial(V)) = n$
excedance of $N = k$.

The following map is well defined:

image of Plücker embedding

$$\Delta(N) : \binom{[n]}{k} \longrightarrow \mathrm{im}(P) \subseteq \mathbb{P}^{\binom{n}{k}-1}$$

$$I \longmapsto \Delta_I(N)$$

such that $\Delta_I(N) = \sum_{\substack{\pi \text{ dimer on } N \\ \partial(\pi) = I}} wt(\pi) = \sum_{\substack{\pi \text{ dimer on } N \\ \partial(\pi) = I}} \prod_{e \in \pi} wt_E(e)$.