Gypsy Akhyar
16th Sep 2025

From Dimers to Webs
Meeting 2

<u>Speaker</u>: James Elhindi

<u>Recap</u>:

(1) Grassmannian of k-planes in n-space

$$Gr(k;n) = \{V \subseteq \mathbb{C}^n \mid V \text{ is a subspace and } \dim(V) = k\}.$$

(2) I- Plücker coordinate

$$P_I(A) = \det \begin{pmatrix} | & & | \\ A_{i_1} & \cdots & A_{i_k} \\ | & & | \end{pmatrix}$$

<u>Example</u> Consider the $n=4$ and $k=2$ example:

$$A = \begin{pmatrix} 2 & 3 & 1 & 6 \\ 9 & 9 & 3 & 0 \end{pmatrix}$$

Then, $P_{\{1,3\}} = \det \begin{pmatrix} 2 & 1 \\ 9 & 3 \end{pmatrix} = 6 - 9 = -3$

Define

$I = \{1\}$	$J = \{1,2,4\}$
$I_1 = \{1,1\}$	$J_1 = \{2,4\}$
$I_2 = \{1,2\}$	$J_2 = \{1,4\}$
$I_3 = \{1,4\}$	$J_3 = \{1,2\}$

Then,

$$P_{I_1} = \det\begin{pmatrix} 2 & 2 \\ 9 & 9 \end{pmatrix} = 0 \qquad P_{J_1} = \det\begin{pmatrix} 3 & 6 \\ 9 & 0 \end{pmatrix} = -54$$

$$P_{I_2} = \det\begin{pmatrix} 2 & 3 \\ 9 & 9 \end{pmatrix} = -9 \qquad P_{J_2} = \det\begin{pmatrix} 2 & 6 \\ 9 & 0 \end{pmatrix} = -54$$

$$P_{I_3} = \det\begin{pmatrix} 2 & 6 \\ 9 & 0 \end{pmatrix} = -54 \qquad P_{J_3} = \det\begin{pmatrix} 2 & 3 \\ 9 & 9 \end{pmatrix} = -9 .$$

Notice that

$$S = (-1)^1 P_{I_1} P_{J_1} + (-1)^2 P_{I_2} P_{J_2} + (-1)^3 P_{I_3} P_{J_3}$$

$$= (-1)(0)(-54) + (1)(-9)(-54) + (-1)(-54)(-9)$$

$$= 0 .$$

Definition (Plücker relations)

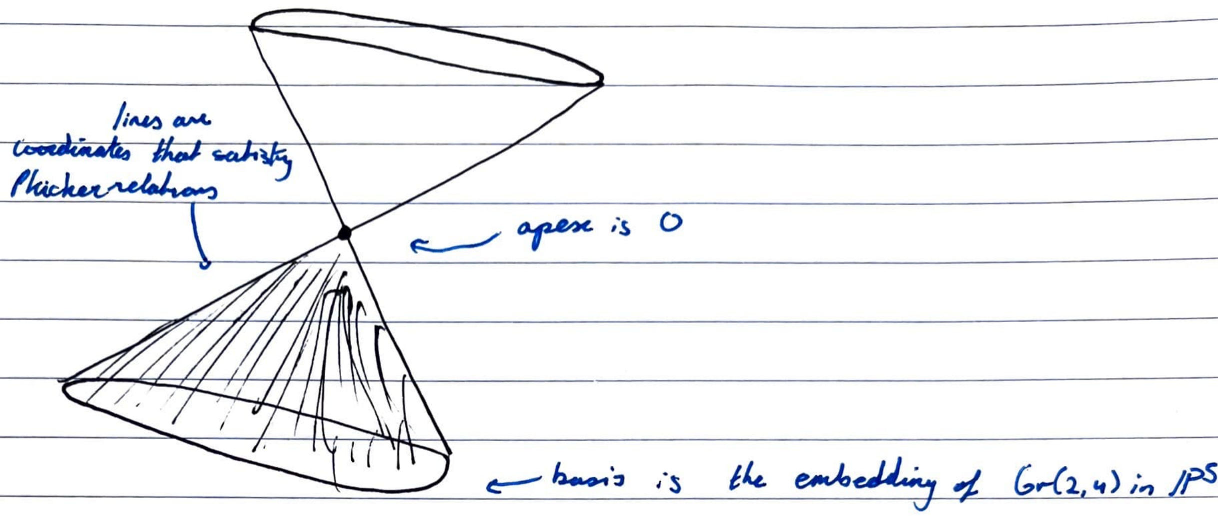
Let

- A be a $k \times n$ matrix
- $P_I(A)$ be the I-Plücker coordinate
- I and J be two ordered sequences: $I = \{i_1, \dots, i_{k-1}\}$, $J = \{j_1, \dots, j_{k+1}\}$ $\subseteq [n]$

Then,

$$\sum_{l=1}^{k+1} (-1)^l P_{\{i_1, \dots, i_{k-1}, j_l\}} P_{\{j_1, \dots, \hat{j_l}, \dots, j_{k+1}\}} = 0$$

Definition (Cone) A cone is the union of all lines that intersect an apex and a basis



Equivalent concepts:

(1) Homogeneous coordinate ring

(2) Coordinate ring of the affine cone

(3) Ring generated by Plücker coordinates with Plücker relations.