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From Dimers to Webs

Meeting 1: Preliminaries on the Grassmannian I

<u>Definition</u> Suppose that $k, n \in \mathbb{Z}_{\geq 0}$ such that $k \leq n$. Then, the <u>Grassmannian of k-planes in n-space</u> is

$$Gr(k;n) = \{ V \subseteq \mathbb{C}^n \mid V \text{ is a subspace and } \dim(V) = k \}$$

For example,

$$Gr(2;3) = \{ 2\text{-planes in } \mathbb{C}^3 \} = \{ \text{planes in } \mathbb{C}^3 \}$$

$$Gr(1;3) = \{ 1\text{-planes in } \mathbb{C}^3 \} = \{ \text{lines in } \mathbb{C}^3 \}$$

$$Gr(24;99) = \{ 24\text{-planes in } \mathbb{C}^{99} \}$$

A special type of Grassmannian is

$$\mathbb{P}^n = Gr(1;n+1) = \{ \text{lines in } \mathbb{C}^{n+1} \}.$$

This is called <u>projective n-space</u>.

The goal of today's talk is to

$$\boxed{\text{tame the Grassmannian}}$$

For us, that means we want to give $Gr(k;n)$ <u>coordinates</u>.

To do this, we will exhibit an injective map

$$Gr(k;n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1} \qquad \text{(Plücker Embedding)}$$

Of course, one might ask:

Why does going to projective space give coordinates?

Really, what you are asking is to

$\boxed{\text{tame projective space}}$.

Proposition The following map is a bijection:

$$\Phi : \mathbb{P}^n \longrightarrow \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^\times}$$

$$V \longmapsto [v], \qquad \text{where } v \in V \setminus \{0\}.$$

Proof.

Recall $\mathbb{P}^n = \{ V \leq \mathbb{C}^{n+1} \mid V \text{ subspace and } \dim(V) = 1 \}$

First, we show that Φ is well-defined.
Suppose $V \in \mathbb{P}^n$ and $v, w \in V \setminus \{0\}$.
Since $\dim(V) = 1$ and $v \in V$, then $V = \text{span}\{v\}$.
So $w \in V = \text{span}\{v\}$ means that there exists $\lambda \in \mathbb{C}^\times$ such that

$$w = \lambda v.$$

So, as elements of $\frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^\times}$,

$$[v] = [w].$$

Next, we show that Φ is injective.
Suppose $V, W \in \mathbb{P}^n$ and $v \in V \setminus \{0\}$, $w \in W \setminus \{0\}$ and $[v] = [w]$.
So, there exists $\lambda \in \mathbb{C}^\times$ such that

$$w = \lambda v .$$

So

$$W = \operatorname{span}\{w\} = \operatorname{span}\{\lambda v\} = \operatorname{span}\{v\} = V$$

Finally, we show that Φ is surjective.
Suppose $[v] \in \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^\times}$.
Then, $\Phi(\operatorname{span}\{v\}) = v$. $\square$

We have done something great - instead of some geometric object

$\operatorname{span}\{(1,2)\} =$  $\in \mathbb{P}^3$,

we can refer to simply a coordinate

$$[(1,2)] \overset{\text{new notation}}{=} [1:2] \in \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^\times}.$$

We have tamed projective space.

Looking at our identification, we observe:

$$\mathbb{P}^n = \frac{\mathbb{C}^{n+1} \setminus \{0\}}{\mathbb{C}^\times}$$

pick a direction of the line.

make it so lines of the same direction are the same.

We now do a similar thing with $Gr(k;n)$.
Define

$$\underline{Mat^\times_{k,n}} = \{ k\times n \text{ matrices with entries in } \mathbb{C} \text{ and are full rank} \}$$
$$= \{ (v_1, \dots, v_k) \mid v_1, \dots, v_k \in \mathbb{C}^n \text{ linearly independent} \}$$

$$\underline{GL_k} = \{ k\times k \text{ invertible matrices with entries in } \mathbb{C} \}$$

and if $A = \begin{pmatrix} — & v_1 & — \\ & \vdots & \\ — & v_k & — \end{pmatrix} \in Mat^\times_{k,n}$, then

$$\underline{rowsp(A)} = span\{v_1, \dots, v_k\}$$

<u>Proposition</u> The following map is a bijection:

$$\Phi : \frac{Mat^\times_{k,n}}{GL_k} \longrightarrow Gr(k;n)$$

$$[A] \longmapsto rowsp(A)$$

<u>Proof</u> is left as an exercise but one should observe:

$$Gr(k;n) = \frac{Mat^\times_{k,n}}{GL_k}$$

pick k directions of the space

make it so directions with the same span (i.e up to a change of basis) are the same.

Definition Suppose that $I=\{i_1<\cdots<i_k\}\subseteq[n]$ and $A\in Mat^{\times}_{k,n}$. The *I-Plücker coordinate of A* is

$$P_I(A)=\det\begin{pmatrix} | & & | \\ A_{i_1} & \cdots & A_{i_k} \\ | & & | \end{pmatrix}$$

where $A=\begin{pmatrix} | & & | \\ A_1 & \cdots & A_n \\ | & & | \end{pmatrix}$.

For example, if $A=\begin{pmatrix} 2 & 3 & 1 & 6 \\ 9 & 9 & 3 & 0 \end{pmatrix}\in Mat^{\times}_{2,4}$, then

$$\begin{aligned} P_{\{1,3\}}(A) &= \det\begin{pmatrix} 2 & 1 \\ 9 & 3 \end{pmatrix} \\ &= 6-9 \\ &= -3 \end{aligned}$$

Theorem (Plücker Embedding) The map

$$\begin{aligned} \Phi: Gr(k;n) &\longrightarrow \mathbb{P}^{\binom{n}{k}-1} \\ V &\longmapsto [P_I(A)]_{I\in\binom{[n]}{k}}, \end{aligned}$$

where $A\in Mat^{\times}_{k,n}$ satisfies $V=\mathrm{rowsp}(A)$, is injective.

Proof

First, we show that Φ is well-defined.
Suppose that $A, B \in Mat_{k,n}^{\times}$ and $V = rowsp(A)$
$= rowsp(B)$

Then, there exists $g \in GL_k$ such that

$$B = gA.$$

Then, if $I \in \binom{[n]}{k}$ we have

$$P_I(B) = P_I(gA)$$

$$= \det \begin{pmatrix} | & & | \\ (gA)_1 & \cdots & (gA)_k \\ | & & | \end{pmatrix}$$

$$= \det \left(g \begin{pmatrix} | & & | \\ A_{i_1} & \cdots & A_{i_k} \\ | & & | \end{pmatrix} \right)$$

$$= \det(g) \cdot P_I(A).$$

Next, we show injectivity.
Suppose $V \in Gr(k;n)$.
We will construct a special matrix $X_V \in Mat_{k,n}^{\times}$ satisfying $V = rowsp(X_W)$.
Start with any matrix $A \in Mat_{k,n}$ satisfying $V = rowsp(A)$.
Since A is full rank, then there exists $I \in \binom{[n]}{k}$ with

$$P_I(A) \neq 0 \quad (\text{exercise})$$

Without loss of generality, suppose $I = [k]$. (exercise)

Define

$$g = \begin{pmatrix} | & & | \\ A_1 & \cdots & A_k \\ | & & | \end{pmatrix}.$$

Since $P_{[k]}(A) \neq 0$, then $g \in GL_k$.

If $I \in \binom{[n]}{k}$, notice

$$\begin{aligned} P_I(A) &= P_I(gg^{-1}A) \\ &= \det(g) \cdot P_I(g^{-1}A) \\ &= P_{[k]}(A) \cdot P_I(g^{-1}A). \end{aligned}$$

We have

$$g^{-1}A = \left(\begin{array}{ccc|ccc} 1 & & & a_{11} & \cdots & a_{1,n-k} \\ & \ddots & & \vdots & \ddots & \vdots \\ & & 1 & a_{k,1} & \cdots & a_{k,n-k} \end{array}\right)$$

where

$$a_{ij} = (-1)^{k-i} P_{([k]\setminus\{i\})\cup\{k+j\}}(A) \quad \text{(exercise)}$$

Define $X_v = (g^{-1}A)(P_{[k]}(A))$

Since $g^{-1} \in GL_k$, then $\mathrm{rowsp}(X_v) = \mathrm{rowsp}(A) = V$.

For the actual injectivity proof, suppose $V, W \in Gr(k;n)$ and construct matrices $X_V, X_W \in Mat^{\times}_{k,n}$ such that $V = \text{rowsp}(X_V)$ as above.
$W = \text{rowsp}(X_W)$

Further, suppose $[P_I(X_V)] = [P_I(X_W)]$.
Write

$$X_V = \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & (a_{ij}) \\ & & 1 & \end{array}\right), \quad X_W = \left(\begin{array}{ccc|c} 1 & & & \\ & \ddots & & (b_{ij}) \\ & & 1 & \end{array}\right)$$

with
$$a_{ij} = (-1)^{k-i} P_{([k]\setminus\{i\})\cup\{k+j\}}(X_V)$$
$$b_{ij} = (-1)^{k-i} P_{([k]\setminus\{i\})\cup\{k+j\}}(X_W)$$

Since $[P_I(X_V)] = [P_I(X_W)]$, then there exists $\lambda \in \mathbb{C}^{\times}$ with

$$P_I(X_W) = \lambda P_I(X_V).$$

So $b_{ij} = \lambda a_{ij}$.

Yet, $P_{[k]}(X_W) = P_{[k]}(X_V) = 1$.

So $\lambda = 1$, so $b_{ij} = a_{ij}$, so $X_V = X_W$, and thus $V = W$.