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Exactly Solvable Models

Exercise 1.5: Show that, alternatively, we can write

$$L_k(u) = \begin{pmatrix} \begin{pmatrix} \sinh(u+\frac{\gamma}{2}) & \\ & \sinh(u-\frac{\gamma}{2}) \end{pmatrix}_k & \begin{pmatrix} & \\ \sinh(\gamma) & \end{pmatrix}_k \\ \begin{pmatrix} & \sinh(\gamma) \\ & \end{pmatrix}_k & \begin{pmatrix} \sinh(u-\frac{\gamma}{2}) & \\ & \sinh(u+\frac{\gamma}{2}) \end{pmatrix}_k \end{pmatrix}_a$$

Notice that:

$$\sinh(u\mathbb{1} + \tfrac{\gamma}{2}\sigma_k^z)$$

when we write this, we mean to say $(\sinh(u\mathbb{1} + \frac{\gamma}{2}\sigma^z))_k$. i.e this operator only acts on the k^{th} site.

$$= \sinh\left(\begin{pmatrix} u+\frac{\gamma}{2} & \\ & u-\frac{\gamma}{2} \end{pmatrix}_k\right)$$

$$= \begin{pmatrix} \sinh(u+\frac{\gamma}{2}) & \\ & \sinh(u-\frac{\gamma}{2}) \end{pmatrix}$$

$$\sigma_k^- \sinh(\gamma)$$

$$= \frac{1}{2}\begin{pmatrix} 0-(0i) & 1-(-i\cdot i) \\ 1-(i\cdot i) & 0-(0i) \end{pmatrix}\sinh(\gamma)$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 1-1 \\ 1+1 & 0 \end{pmatrix}\sinh(\gamma)$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\sinh(\gamma)$$

$$= \begin{pmatrix} 0 & 0 \\ \sinh(\gamma) & 0 \end{pmatrix}$$

$\sigma_{k}^{+} \sinh(\gamma)$

$= \frac{1}{2}\begin{pmatrix} 0+(0i) & 1+(-i\cdot i) \\ 1+(i\cdot i) & 0+0i \end{pmatrix} \sinh(\gamma)$

$= \frac{1}{2}\begin{pmatrix} 0+0 & 1+1 \\ 1-1 & 0+0 \end{pmatrix} \sinh(\gamma)$

$= \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \sinh(\gamma)$

$= \begin{pmatrix} 0 & \sinh(\gamma) \\ 0 & 0 \end{pmatrix}$

$\sinh(u\mathbb{1} - \frac{\gamma}{2}\sigma_{k}^{z})$

$= \sinh\begin{pmatrix} u-\frac{\gamma}{2} & \\ & u+\frac{\gamma}{2} \end{pmatrix}$

$= \begin{pmatrix} \sinh(u-\frac{\gamma}{2}) & \\ & \sinh(u+\frac{\gamma}{2}) \end{pmatrix}$

as required.

Exercise 1.6: Explicitly compute $T(u)$ when $N=2$.

$T(u) = L_1(u)\, L_2(u)$

$= \begin{pmatrix} \sinh(u\mathbb{1}+\frac{\gamma}{2}\sigma_1^z) & \sigma_1^- \sinh(\gamma) \\ \sigma_1^+ \sinh(\gamma) & \sinh(u\mathbb{1}-\frac{\gamma}{2}\sigma_1^z) \end{pmatrix} \cdot \begin{pmatrix} \sinh(u\mathbb{1}+\frac{\gamma}{2}\sigma_2^z) & \sigma_2^- \sinh(\gamma) \\ \sigma_2^+ \sinh(\gamma) & \sinh(u\mathbb{1}-\frac{\gamma}{2}\sigma_2^z) \end{pmatrix}$

$= \begin{pmatrix} \sinh(u\mathbb{1}+\frac{\gamma}{2}\sigma_1^z)\sinh(u\mathbb{1}+\frac{\gamma}{2}\sigma_2^z) + \sigma_1^-\sigma_2^+\sinh^2(\gamma) & \sigma_2^- \sinh(u\mathbb{1}+\frac{\gamma}{2}\sigma_1^z)\sinh(\gamma) + \sigma_1^- \sinh(u\mathbb{1}-\frac{\gamma}{2}\sigma_2^z)\sinh(\gamma) \\ \sigma_1^+ \sinh(u\mathbb{1}+\frac{\gamma}{2}\sigma_2^z)\sinh(\gamma) + \sigma_2^+ \sinh(u\mathbb{1}-\frac{\gamma}{2}\sigma_1^z)\sinh(\gamma) & \sigma_1^+\sigma_2^- \sinh^2(\gamma) + \sinh(u\mathbb{1}-\frac{\gamma}{2}\sigma_1^z)\sinh(u\mathbb{1}-\frac{\gamma}{2}\sigma_2^z) \end{pmatrix}$