Gyorgy Akhyer
28th July 2025

Exactly Solvable Models

Exercise 1.3: Compute explicitly the Basis($V$) when $n=3$.

We have:

$$\text{Basis}(V) = \{ |i_1, i_2, i_3\rangle \mid i_k \in \{\pm\} \}$$
$$= \begin{Bmatrix} |-,-,-\rangle, & |-,-,+\rangle, & |-,+,-\rangle \\ |-,+,+\rangle, & |+,-,-\rangle, & |+,-,+\rangle \\ |+,+,-\rangle, & |+,+,+\rangle & \end{Bmatrix}$$

Exercise 1.4: Explicitly compute $H$ for $N=3$.

We have:

$$H = \sum_{k=1}^{3} \left( J^x \sigma_k^x \sigma_{k+1}^x + J^y \sigma_k^y \sigma_{k+1}^y + J^z \sigma_k^z \sigma_{k+1}^z \right)$$
$$= J^x(\sigma_1^x\sigma_2^x + \sigma_2^x\sigma_3^x + \sigma_3^x\sigma_1^x) + J^y(\sigma_1^y\sigma_2^y + \sigma_2^y\sigma_3^y + \sigma_3^y\sigma_1^y)$$
$$+ J^z(\sigma_1^z\sigma_2^z + \sigma_2^z\sigma_3^z + \sigma_3^z\sigma_1^z)$$

$$= J^x(\sigma^x \otimes \sigma^x \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^x \otimes \sigma^x + \sigma^x \otimes \mathbb{1} \otimes \sigma^x)$$
$$+ J^y(\sigma^y \otimes \sigma^y \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^y \otimes \sigma^y + \sigma^y \otimes \mathbb{1} \otimes \sigma^y)$$
$$+ J^z(\sigma^z \otimes \sigma^z \otimes \mathbb{1} + \mathbb{1} \otimes \sigma^z \otimes \sigma^z + \sigma^z \otimes \mathbb{1} \otimes \sigma^z)$$

Now, $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

So, $\sigma^x \otimes \sigma^x = \begin{pmatrix} 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$, $\sigma^x \otimes \mathbb{1} = \begin{pmatrix} 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ 0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & 1\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

So $\sigma^x \otimes \sigma^z \otimes \mathbb{1} = \begin{pmatrix} 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 1\begin{pmatrix}1&0\\0&1\end{pmatrix} \\ 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 1\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} \\ 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 1\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} \\ 1\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} & 0\begin{pmatrix}1&0\\0&1\end{pmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 1&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0 \end{pmatrix}$$

Computing the rest on mathematica, we obtain:

$$H = \begin{pmatrix} 3J^z & 0 & 0 & J^x-J^y & 0 & J^x-J^y & J^x-J^y & 0 \\ 0 & -J^z & J^x+J^y & 0 & J^x+J^y & 0 & 0 & J^x-J^y \\ 0 & J^x+J^y & -J^z & 0 & J^x+J^y & 0 & 0 & J^x-J^y \\ J^x-J^y & 0 & 0 & -J^z & 0 & J^x+J^y & J^x+J^y & 0 \\ 0 & J^x+J^y & J^x+J^y & 0 & -J^z & 0 & 0 & J^x-J^y \\ J^x-J^y & 0 & 0 & J^x+J^y & 0 & -J^z & J^x+J^y & 0 \\ J^x-J^y & 0 & 0 & J^x+J^y & 0 & J^x+J^y & -J^z & 0 \\ 0 & J^x-J^y & J^x-J^y & 0 & J^x-J^y & 0 & 0 & 3J^z \end{pmatrix}$$