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Exactly Solvable Models
Lecture 35

(Non-Examinable)

5. Fermions, Tau Functions and Integrable hierarchies

These are infinite sets of nonlinear PDEs that are simultaneously solvable.

$\underbrace{\text{KdV hierarchy (1D)} \;\}\; \text{KP hierarchy (2D)} \;\}\; \text{Toda hierarchy (Toda lattice)}}_{\text{Examples.}}$

Our approach to studying them is due to the Kyoto school (early 1980s).

This uses the language of fermions (wedge product spaces).

§1. Fermions and the Clifford algebra

Introduce the set

$$S = \{\psi_i\}_{i\in\mathbb{Z}} \cup \{\psi_i^*\}_{i\in\mathbb{Z}}$$

whose elements are called fermions.

We assign these elements a charge

$$ch(\psi_i) = 1 \qquad \text{and} \qquad ch(\psi_i^*) = -1$$

Construct words

$$W = x_1 \cdots x_m, \quad x_k \in S, \quad m \in \mathbb{Z}_{\geq 0}.$$

The Clifford algebra is defined as the set

$$\left\{ \sum_W c_W W \;\middle|\; W \text{ is a word}, \; c_W \in \mathbb{C} \right\} = Cl$$

with relations

$$[\psi_i, \psi_j]_+ = 0 \qquad [\psi_i^*, \psi_j^*]_+ = 0$$

$$[\psi_i, \psi_j^*]_+ = \delta_{ij} \qquad ([A, B]_+ = AB + BA).$$

Define the charge of a word $W = x_1 \cdots x_m$ to be

$$ch(W) = \sum_{k=1}^{m} ch(x_k)$$

We can consider a grading on Cl:

$$Cl = \bigoplus_{i \in \mathbb{Z}} Cl^{(i)} \quad \text{where} \quad Cl^{(i)} = \left\{ \sum_W c_W W \;\middle|\; ch(W) = i \right\}$$

§2. Fock representation

Introduce the 'vacuum' vector $|\emptyset\rangle$ and define the following actions:

$\psi_m |\emptyset\rangle = 0, \quad m < 0$ (trying to place a particle at position m)

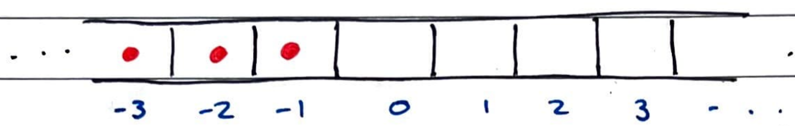
$\psi_n^* |\emptyset\rangle = 0, \quad n \geq 0$ (trying to remove a particle at position n)

To understand these particle placing/removing analogy, we associate a string

$$(\ldots, S_{-2}, S_{-1}, S_0, S_1, S_2, \ldots)$$

to $|\emptyset\rangle$ such that (a) $S_{-k}=1$ for all $k \geq 1$
(b) $S_k = 0$ for all $k \geq 0$

Draw:



This suggests a general rule for the action of the fermions on any state

$$|S\rangle = |\ldots, S_{-2}, S_{-1}, S_0, S_1, S_2, \ldots$$

where $S_{-k} = 1$ and $S_k = 0$ for all k sufficiently large.

We have,

$$\psi_m |S\rangle = \begin{cases} |\ldots, S_m + 1, \ldots\rangle (-1)^{\sum_{j<m}(1-S_j)}, & S_m = 0 \\ 0, & S_m \neq 0 \end{cases}$$

$$\psi_n^* |S\rangle = \begin{cases} |\ldots, S_n - 1, \ldots\rangle (-1)^{\sum_{j<n}(1-S_j)}, & S_n = 1 \\ 0, & S_n \neq 1 \end{cases}$$

We denote this representation space

$$\mathcal{F} = \mathcal{Cl}\,|\emptyset\rangle = \left\{ \sum_S c_S |S\rangle \right\}$$

The Fock space splits up into a direct sum of subspaces:

$$\mathcal{F} = \bigoplus_{k \in \mathbb{Z}} \underbrace{\mathcal{F}^{(k)}}_{\text{charged subspaces}} , \qquad \mathcal{F}^{(k)} = \mathcal{A}^{(k)} |\emptyset\rangle = \left\{ \sum_{S} c_S |S\rangle \;\middle|\; ch(S) = k \right\} .$$

where the <u>charge of a string</u> is:

$$ch(S) = \sum_{k \geq 0} S_k + \sum_{k < 0} (S_k - 1) .$$

§3. Partition Basis

For any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_l \geq 0)$, we will define

$$\underline{|\lambda\rangle} = \psi_{\lambda_1 - 1} \cdots \psi_{\lambda_l - l} \psi^*_{-l} \cdots \psi^*_{-1} |\emptyset\rangle$$

Fermionic representation of a partition.

Then, one may show that

$$\mathcal{F}^{(0)} = \left\{ \sum_{\lambda} c_\lambda |\lambda\rangle \right\} .$$

We call this the <u>partition basis</u>.