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24th Oct 2025

Exactly Solvable Models
Lecture 34

Ascending Schur Measure (i.e. $x_i = 1$ and $\vec{z}$ is Plancherel specialised)

$$P_{\text{planch}}(\lambda^{(1)}, \ldots, \lambda^{(N)}) = \mathbb{1}_{\lambda^{(1)} \prec \cdots \prec \lambda^{(N)}} \, S_{\lambda^{(N)}}(Pl_t)\, e^{-Nt}$$

Joint probability distribution of GUE

$$P_{GUE}(\{\theta_i^{(j)}\}) = \mathbb{1}_{\theta^{(1)} \prec \cdots \prec \theta^{(N)}} \left(\frac{1}{2\pi}\right)^{N/2} \prod_{1 \le i < j \le N} (\theta_i^{(N)} - \theta_j^{(N)}) \prod_{i=1}^{N} e^{-\frac{1}{2}\theta_i^{(N)2}} \, d\theta,$$

where $d\theta = \prod_{1 \le i \le j \le N} d\theta_i^{(j)}$ is the product measure,

largest eigenvalue of $j \times j$ block

$$\theta^{(j)} = (\theta_1^{(j)} \ge \cdots \ge \theta_j^{(j)})$$

interlacing means $\theta_i^{(j)} \ge \theta_i^{(j-1)} \ge \theta_{i+1}^{(j)}$ for all $1 \le i \le j-1$.

Theorem (Vershik) Let $\lambda_i^{(j)}$ denote the i^{th} part of the partition $\lambda^{(j)}$ in the process $M_t^{\hat{}}$. Define

$$\tilde{\lambda}_i^{(j)} = \frac{\lambda_i^{(j)} - t}{t^{1/2}}.$$

Then, the following equivalence holds:

$$\left\{\lim_{t\to\infty} \tilde{\lambda}_i^{(j)}\right\}_{1 \le i \le j \le N} \overset{d}{=} \left\{\theta_i^{(j)}\right\}_{1 \le i \le j \le N}$$

Proof (sketch)

Make the substitution $\lambda_i^{(j)} = t + t^{\frac{1}{2}} \theta_i^{(j)}$ where $\theta_i^{(j)} \in \mathbb{R}$ and study the limit as $t \to \infty$. We want to show that

$$\mathbb{P}_{\text{planch}} \to \mathbb{P}_{\text{GUE}}$$

in this regime.

First, note that $\lambda^{(j)} \succ \lambda^{(j-1)}$. Then, it is easy to show that this implies $\theta^{(j)} \succ \theta^{(j-1)}$.

Next (writing $\lambda^{(N)} = \lambda$), we wish to show that

$$\underset{\substack{\uparrow \\ N \text{ parts}}}{S_\lambda}(\rho_{t})e^{-Nt} \sim \underset{\substack{\nearrow \\ \text{Jacobian from} \\ \text{variable change}}}{t^{-\binom{N+1}{2}\frac{1}{2}}} \left(\frac{1}{2\pi}\right)^{N/2} \prod_{1 \le i < j \le N} (\theta_i - \theta_j) \prod_{i=1}^{N} e^{-\frac{1}{2}\theta_i^2}$$

Now, to do asymptotics, it would be nice to have an integral formula.

Proposition Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be a partition with $\ell(\lambda) \le N$. Then,

$$S_\lambda(z_1, \ldots, z_n) = \oint \frac{dw_1}{2\pi i} \cdots \oint \frac{dw_N}{2\pi i} \prod_{1 \le i < j \le N} (w_j - w_i) \prod_{i=1}^{N} w_i^{\lambda_i - i + N} \prod_{i=1}^{n} \prod_{j=1}^{N} \frac{1}{w_i - z_j}$$

where C is a positively oriented circle centered on the origin that encloses $\{z_1, \ldots, z_n\}$.

brute force computation of residues.

After performing the N integrals explicitly (YOU SHOULD DO THIS!),

$$S_\lambda(z_1, \dots z_n) = \sum_{\sigma \in S_n} \prod_{1 \le i < j \le n} \frac{z_{\sigma(i)}}{z_{\sigma(i)} - z_{\sigma(j)}} \prod_{i=1}^{n} z_{\sigma(i)}^{\lambda_i}$$

$$\overset{\text{(Laplace Expansion)}}{=} \frac{\det_{1 \le i, j \le n} z_i^{\lambda_j - j + n}}{\prod_{1 \le i < j \le n} z_i - z_j}$$

symmetrisation formula (discrete)

integral formula (analytic)

we can now use calculus tools!

One has

$$S_\lambda(\rho|_t) = \oint_C \frac{dw_1}{2\pi i} \cdots \oint_C \frac{dw_N}{2\pi i} \prod_{1 \le i < j \le N} (w_j - w_i) \prod_{i=1}^{N} w_i^{\lambda_i - i} \cdot e^{t/w_i}$$

$$= \oint_C \frac{dw_1}{2\pi i} \cdots \oint_C \frac{dw_N}{2\pi i} \prod_{1 \le i < j \le N} (w_j - w_i) \prod_{i=1}^{N} w_i^{-i} \prod_{i=1}^{N} e^{(t + t^{\frac{1}{2}}\theta_i)\log w_i + t/w_i}$$

after letting $\lambda_i = t + t^{\frac{1}{2}}\theta_i$ for each $i \in [1, N]$.

One then computes the critical point of the function

$$\log(w) + \frac{1}{w},$$

which is the value where the derivative vanishes:

$$\frac{1}{w_c} - \frac{1}{w_c^2} = 0 \implies w_c = 1$$

About the point $w=1$, one has the Taylor series:

$$t\left(\log(w) + \frac{1}{w}\right) = t\left(1 + \frac{1}{2}(w-1)^2 + O(w-1)^3\right).$$

Deforming all contours to pass through $w_i = 1$, the dominant contribution to the asymptotic comes by a small line segment passing through $w_i = 1$, and along which the imaginary part of $\frac{1}{2}(w_i-1)^2$ is constant (saddle point analysis).

This yields, after switching variables $w_i = 1 + z_i t^{-\frac{1}{2}}$,

$$S_\lambda(Pl_t) \sim \left(\frac{1}{2\pi i}\right)^N t^{-\left(\binom{N+1}{2}\right)/2} e^{Nt} \int_{-i\infty}^{i\infty} dz_1 \cdots \int_{-i\infty}^{i\infty} dz_N \prod_{1\le i<j\le N} (z_j - z_i)$$
$$\prod_{i=1}^{N} \exp\left(\frac{z_i^2}{2} + z_i \theta_i\right)$$

as $t \to \infty$.