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Exactly Solvable Models
Lecture 32

§10. Limit to the GUE corners process

Recap

The ascending Schur measure

$$\mathbb{P}_{asc}(\lambda^{(1)},\dots,\lambda^{(n)}) = \left[\prod_{i=1}^{N} s_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i)\right] s_{\lambda^{(N)}}(z_1,\dots,z_n)\left[\prod_{i=1}^{N}\prod_{j=1}^{n}(1-x_i z_j)\right].$$

We extend this to a process

$$\emptyset \xrightarrow{M^{\uparrow}_{x_1;\vec{z}}} \lambda^{(1)} \longrightarrow \cdots \xrightarrow{M^{\uparrow}_{x_N;\vec{z}}} \lambda^{(n)}$$

where

$$M^{\uparrow}_{x;\vec{z}}(\mu\to\lambda) = s_{\lambda/\mu}(x)\cdot \frac{s_\lambda(\vec{z})}{s_\mu(\vec{z})}\prod_{j=1}^{n}(1-xz_j).$$

Definition (Plancherel specialisation)

Given a symmetric polynomial $P(z_1,\dots,z_n)$, define

$$P(Pl_t) = \lim_{n\to\infty} P(t/n,\dots,t/n).$$

Some facts about Pl_t:

(1) Define the power sum symmetric polynomials as

$$p_k(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i^k$$

$$= z_1^k + \cdots + z_n^k$$

We have

$$p_k(Pl_t) = \begin{cases} t, & k = 1 \\ 0, & \text{otherwise.} \end{cases}$$

(2) Specialising the z parameters in the Cauchy kernel gives

$$\left(\prod_{i=1}^{N} \prod_{j=1}^{n} \frac{1}{1 - x_i z_j} \right) \Bigg|_{Pl_t} = \lim_{n \to \infty} \prod_{i=1}^{N} \left(1 - x_i t/n\right)^{-n}$$

$$= \prod_{i=1}^{n} e^{-x_i t}.$$

Define the Plancherel specialisation of the Schur measure

$$\mathbb{P}_{\text{plancherel}}(\lambda^{(1)}, \ldots, \lambda^{(n)})$$

$$= \mathbb{P}_{\text{Asc}}(\lambda^{(1)}, \ldots, \lambda^{(n)}) \Big|_{\substack{x_i = 1 \\ (z_1, \ldots, z_n) | Pl_t}} = \mathbb{1}_{\lambda^{(1)} \prec \cdots \prec \lambda^{(N)}} \cdot s_{\lambda^{(N)}}(Pl_t) \cdot e^{-Nt}$$

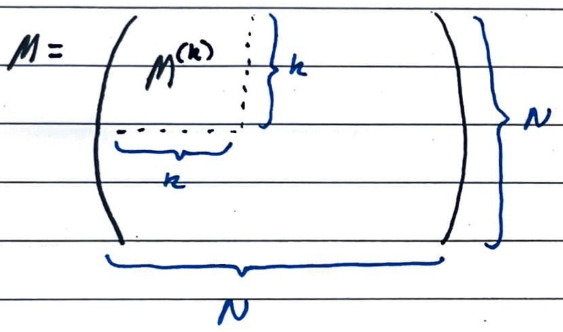
Q: What is the $t \to \infty$ asymptotic behaviour of $\mathbb{P}_{\text{plancherel}}$?

Definition (Gaussian Unitary Ensemble (GUE))

Construct the set of all $N \times N$ Hermitian ($M = (M^T)^*$) matrices whose real and imaginary parts are i.i.d. Gaussian random variables (taken from $N(0,1)$).

By construction, this set has the structure of a probability space.

Given $M \in GUE$, we let $M^{(k)}$ denote the top-left corner of M.



Let $\theta_1^{(k)} \leq \dots \leq \theta_k^{(k)}$ denote the eigenvalues of $M^{(k)}$ and $\theta^{(k)} = (\theta_k^{(k)}, \dots, \theta_{(1)}^{(k)})$.

Definition The GUE corners process is the joint law of the eigenvalues $\theta_i^{(j)}$ for $1 \leq i \leq j \leq N$.

(top index is the block, bottom index is its relative size)

Theorem The eigenvalues of the GUE corners process are given by the probability density function

$$p_{GUE}(\{\theta_i^{(j)}\}) = \mathbb{1}_{\theta^{(1)} < \dots < \theta^{(N)}} \left(\prod_{1 \leq i < j \leq N} \theta_j^{(N)} - \theta_i^{(N)} \right) \left(\prod_{i=1}^{N} e^{-\frac{1}{2} \theta_i^{(N)2}} \right) \left(\frac{1}{2\pi} \right)^{N/2}.$$