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Exactly Solvable Models
Lecture 31

§9. Schur Processes

What's coming up:
- Schur measures
- Markov dynamics
- Plane partitions
- Large time limit of the process
- Gaussian Unitary Ensemble

Schur measure

We have seen previously that $G_{S/T}$ (in the 6-vertex model) defines a random string S (given T).

We would like an analogue of this for our new functions $G_{S/T}$ (in our 'other' 6-vertex model). However, these weights are not stochastic.

This motivates the introduction of the Schur measures.
There are three types:

(Ascending Schur Measure)

$$P(\varnothing \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(N)}) = \left[\prod_{i=1}^{N} s_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i)\right] s_{\lambda^{(N)}}(z_1,\dots,z_n) \cdot \prod_{i=1}^{N}\prod_{j=1}^{n}(1-x_i z_j)$$

(Measure on space of all partitions)

$$P(\lambda) = s_\lambda(x_1,\dots,x_N)\, s_\lambda(z_1,\dots,z_n) \prod_{i=1}^{N}\prod_{j=1}^{n}(1-x_i z_j)$$

(Plane partition measure)

$$\mathbb{P}(\emptyset \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(N)} \succ \cdots \succ \lambda^{(N+n)} \succ \emptyset)$$

$$= \prod_{i=1}^{N} s_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i) \prod_{j=1}^{n} s_{\lambda^{(N+j)}/\lambda^{(N+j+1)}}(z_j) \prod_{i=1}^{N} \prod_{j=1}^{n} (1 - x_i z_j)$$

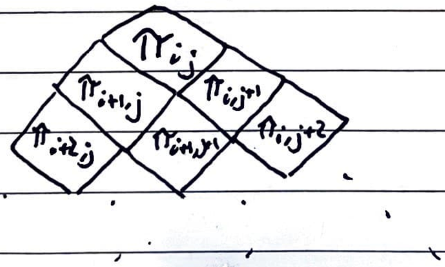
There is a one-to-one correspondence between sequences of the above form and plane partitions.

Definition A plane partition $\pi = \{\pi_{i,j}\}_{i,j \geq 1} \subseteq \mathbb{Z}_{\geq 0}$ satisfies two defining properties:

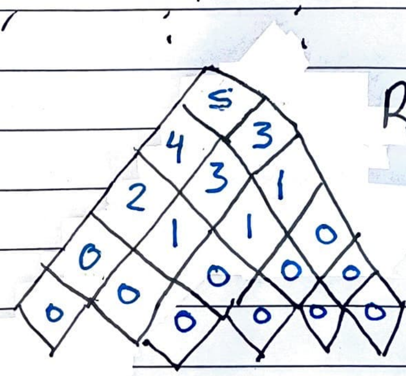
(1) If i,j are sufficiently large, then $\pi_{i,j} = 0$.

(2) (a) $\pi_{i,j} \geq \pi_{i,j+1}$
(b) $\pi_{i,j} \geq \pi_{i+1,j}$ $\Bigg\} \longrightarrow$ $\begin{array}{ccc} & \pi_{i,j} & \\ \rotatebox{90}{$\leq$} & & \rotatebox{-45}{$\geq$} \\ \pi_{i+1,j} & & \pi_{i,j+1} \end{array}$

In this way, we illustrate plane partitions as follows:



For example:



Reading the diagonals (verticals) we get:

$$\emptyset \prec (2) \prec (4\ 1) \prec (5\ 3) \succ (3\ 1) \succ (1) \succ \emptyset$$

Or, $\emptyset \prec$ ⬚⬚ $\prec$ ⬚⬚⬚⬚ $\prec$ ⬚⬚⬚⬚ $\succ$ ⬚⬚ $\succ$ ⬚ $\succ \emptyset$.

There is a nice degeneration of the plane partition measure.

Recall $S_{\lambda/\mu}(x) = \begin{cases} x^{|\lambda|-|\mu|} & , \lambda \prec \mu \\ 0 & , \text{otherwise} \end{cases}$

Using this fact, along with the specialisation

$$\left.\begin{aligned} x_N &= q^{\frac{1}{2}} & z_1 &= q^{\frac{1}{2}} \\ &\vdots & &\vdots \\ x_{N-1} &= q^{3/2} & z_2 &= q^{3/2} \\ &\vdots & &\vdots \\ x_i &= q^{N-i+\frac{1}{2}} & z_i &= q^{j-\frac{1}{2}} \\ &\vdots & &\vdots \\ x_1 &= q^{N-\frac{3}{2}} & z_n &= q^{n-\frac{1}{2}} \end{aligned}\right\}$$

this is the principal specialisation.

the offset is OK as Schur polynomials are homogeneous.

One may verify that, under this choice

$$\underline{\mathbb{P}_{pp}(\pi)} = q^{|\pi|} \prod_{i=1}^{N} \prod_{j=1}^{n} (1 - q^{i+j-1})$$

where $|\pi| = \sum_{i,j} \pi_{i,j}$.

This matches with a known generating series for plane partitions:

$$\sum_{\pi \in N \times n \text{ box}} q^{|\pi|} = \frac{1}{\prod_{i=1}^{N} \prod_{j=1}^{n} (1-q^{i+j-1})}$$

Taking n, N to infinity,

$$\sum_{\pi} q^{|\pi|} = \frac{1}{\prod_{i=1}^{\infty} (1-q^i)^i}$$

The ascending Schur measure and plane partition measure have a connection with Markov dynamics.

Definition

$$\mathbb{M}^{\uparrow}_{x;\vec{z}}(\mu \to \lambda) = s_{\lambda/\mu}(x) \cdot \frac{s_\lambda(\vec{z})}{s_\mu(\vec{z})} \cdot \prod_{j=1}^{n} (1 - x z_j)$$

(One should check that, via the skew Cauchy identity, that $\sum_{\lambda} \mathbb{M}_{x;z}(\mu \to \lambda) = 1$)

Let $y \in \{z_1, z_2, \ldots\}$.

$$\mathbb{M}^{\downarrow}_{y;\vec{z}}(\lambda \to \mu) = s_{\lambda/\mu}(y) \cdot \frac{s_\mu(\vec{z} \setminus \{y\})}{s_\lambda(\vec{z})}$$

(One should check that, via the branching rule, that $\sum_{\mu} \mathbb{M}^{\downarrow}_{y;\vec{z}}(\lambda \to \mu) = 1$)