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Exactly Solvable Models
Lecture 30

Recap We found

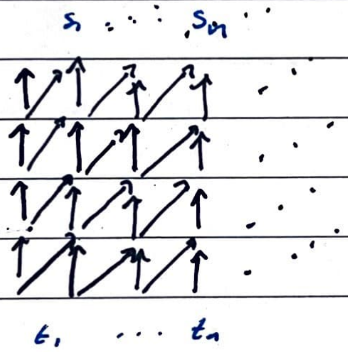
$$Z(\{t_1, \dots, t_n\} \to \{s_1, \dots, s_n\}) = \det_{1 \le i,j \le n} \left(h_{s_i - t_j} \right)$$

$$= \det_{1 \le i,j \le n} \left(h_{\lambda_i - i - \mu_j + j} \right)$$

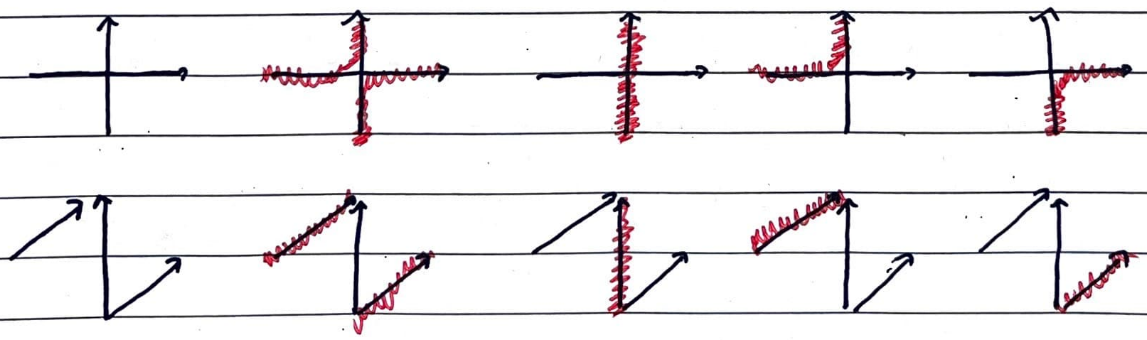
$\left(\begin{array}{l} \text{noting that} \\ s_i = \lambda_{n-i+1} + i \\ t_i = \mu_{n-i+1} + i \end{array} \right)$

$$= S_{\lambda/\mu}$$

A similar argument works on this graph

$g =$ 

where we utilise the following bijection of the $x=0$ weights:



Then, the multipath PF is given by

$$Z(\{t_1, \dots, t_n\} \to \{s_1, \dots, s_n\}) = \det_{1 \le i,j \le n} \left(e_{s_i - t_j} \right)$$

$$= \det_{1 \le i,j \le n} \left(e_{\lambda_i' - i - \mu_j' + j} \right)$$

$$= s_{\lambda'/\mu'} .$$

Exercise Show that the following reduction holds:

$$F_S(z_1, \dots, z_n; 0) = \prod_{i=1}^{n} z_i^{-1} \, s_\lambda(z_1, \dots, z_n), \qquad \lambda = \rho(S) .$$

§8. Hidden symmetry

It turns out that both $G_{S/T}$ and F_S have further symmetries in their alphabets.

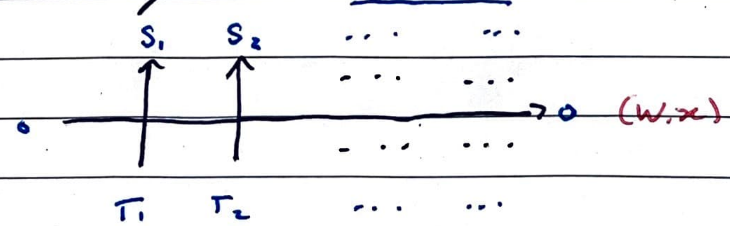
Theorem The function $G_{S/T}(x_1, \dots, x_N, w_1, \dots, w_N)$ is symmetric in x and w separately.

(Recall previously we had $(x_i, w_i) \leftrightarrow (x_j, w_j)$)

Proof.

The proof hinges on a property of the model.
Define the monodromy matrix operator $A(w,x)$:

$$\langle T | A(w,x) | S \rangle =$$



The following holds (concatenation lemma):

$$A(w,x) = A(w,0) \cdot A(0,x) .$$

As a consequence of this, we have

$$G_{S/T}(x_1,\dots,x_N;w_1,\dots,w_N)$$

$$=\langle T|A(w_1,x_1)\cdots A(w_N,x_N)|S\rangle$$

$$=\langle T|A(w_1,0)\cdots A(w_N,0)A(0,x_1)\cdots A(0,x_N)|S\rangle \quad \dots (1)$$

$$=\langle T|A(0,x_1)\cdots A(0,x_N)A(w_1,0)\cdots A(w_N,0)|S\rangle \quad \dots (2)$$

(1) is equal to

$$\sum_{\nu} S_{\nu'/\mu'}(w_1,\dots,w_N)\, S_{\lambda/\nu}(x_1,\dots,x_N)$$

(2) is equal to

$$\sum_{\nu} S_{\nu/\mu}(x_1,\dots,x_N)\cdot S_{\lambda'/\nu'}(w_1,\dots,w_N)$$

From these, the claimed symmetry holds.

[Macdonald, Chapter I, Section 5, Example 23].