Exactly Solvable Models
Lecture 28

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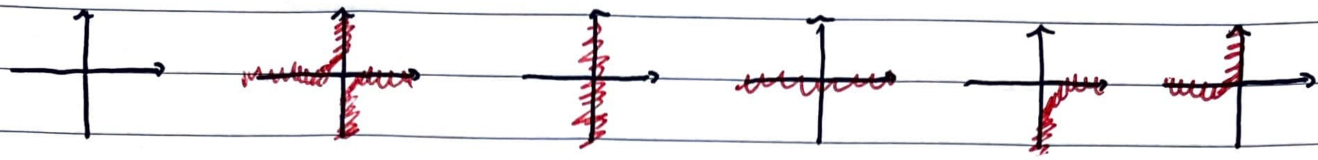
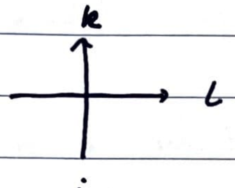
Another 6-vertex model



1   w 1 x $x+w$ 1

where the generic vertex is

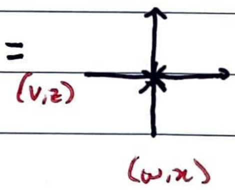
$L_{bc}(w,x) = (w,x)_{j}$ 

, $i,j,k,l \in \{0,1\}$

Theorem There exists an intertwiner R_{ab} such that

$$R_{ab}\, L_{ac}(v,z)\, L_{bc}(w,x) = L_{bc}(w,x)\, L_{ac}(v,z)\, R_{ab}$$

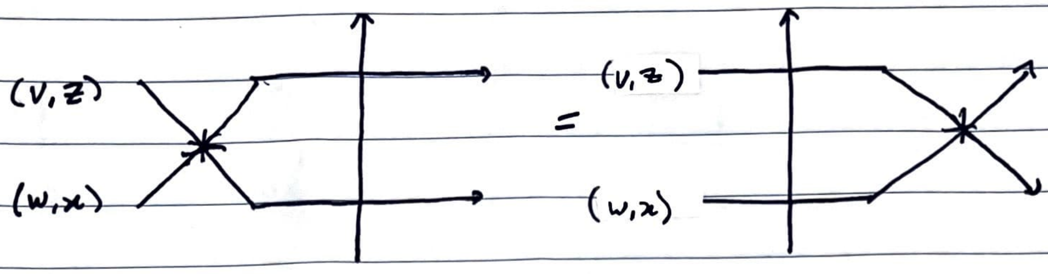
where   note the new normalisation!

$$R_{ab} = \begin{pmatrix} 1 & & & \\ & \frac{w-v}{w+z} & \frac{v+z}{w+z} & \\ & \frac{w+x}{w+z} & \frac{z-x}{w+z} & \\ & & & \frac{v+x}{w+z} \end{pmatrix}_{ab} =$$ 

$\in \mathrm{End}(V_a \otimes V_b)$

Proof by direct calculation.

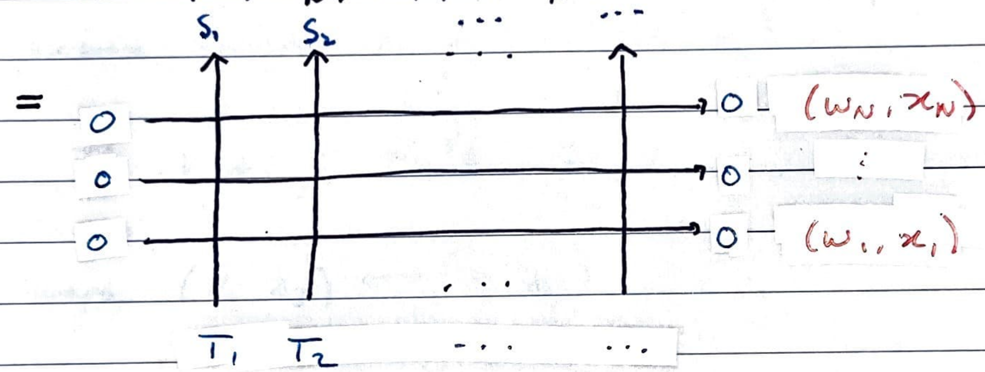
Note that we can write this RLL equation graphically:



§6. Functions $G_{S/T}$ and F_S

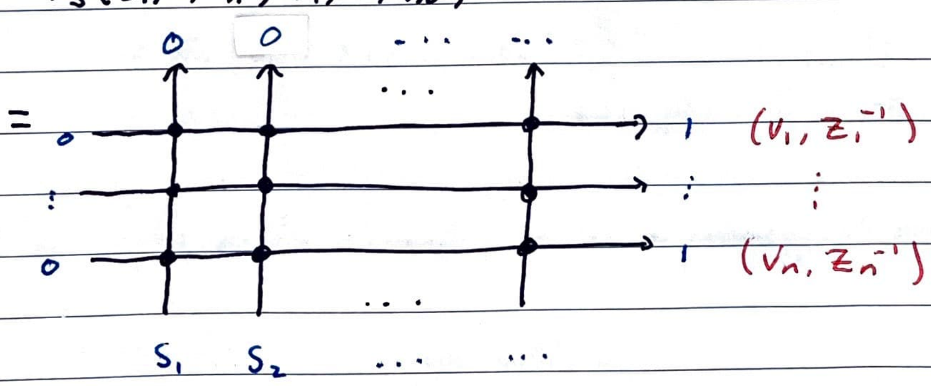
Using our new vertex model, we define two families of polynomials:

$G_{S/T}(x_1, \dots, x_N; w_1, \dots, w_N)$

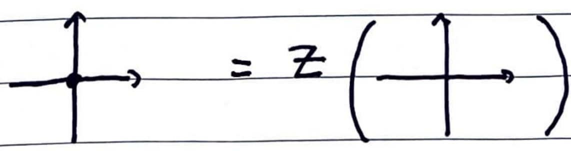


and

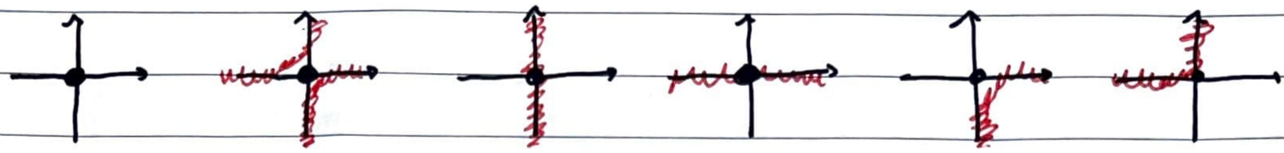
$F_S(z_1, \dots, z_n; v_1, \dots, v_n)$



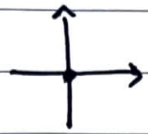
where



The tabulation of the dotted weights are as follows:



$Z \qquad VZ \qquad Z \qquad 1 \qquad 1+VZ \qquad Z$

for the generic weight  (v, Z)

<u>Theorem</u>

<u>(1)</u> $G_{S/T}(x_1, \dots, x_N; w_1, \dots, w_N)$ is a symmetric polynomial under pairwise swaps $(x_i, w_i) \leftrightarrow (x_j, w_j)$.

<u>(2)</u> $\prod_{1 \le i < j \le n} (v_i + z_j)^{-1} F_S(z_1, \dots, z_n; v_1, \dots, v_n)$ is symmetric under swaps $(v_i, z_i) \leftrightarrow (v_j, z_j)$.

<u>(3)</u> The functions satisfy a Cauchy identity

$$\sum_S G_{S/T}(x_1, \dots x_N; w_1, \dots, w_N) \, F_S(z_1, \dots, z_n, v_1, \dots, v_n)$$

$$= F_T(z_1, \dots, z_n; v_1, \dots, v_n) \prod_{i=1}^{N} \prod_{j=1}^{n} \frac{1 + w_i z_j}{1 - x_i z_j}.$$

← Cauchy kernel

This holds either formally, or as a convergent sum assuming

$|x_i z_j| < 1$ for all i, j

Remark The Cauchy kernel $\displaystyle\prod_{i,j} \frac{1+w_i z_j}{1-x_i z_j}$ contains both the kernel for the ordinary Cauchy identity:

$$\sum_\lambda s_{\lambda/\mu}(x)\, s_\lambda(y) = s_\mu(y) \prod_{i,j} \frac{1}{1-x_i y_j}$$

and the dual Cauchy identity:

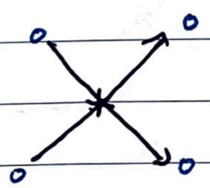
$$\sum_\lambda s_{\lambda'/\mu'}(x)\, s_\lambda(y) = s_\mu(y) \prod_{i,j} (1+x_i y_j)$$

as special cases. This suggests that there are two degenerations:

$$G_{S/T} \nearrow s_{\lambda/\mu} \qquad G_{S/T} \searrow s_{\lambda'/\mu'}$$

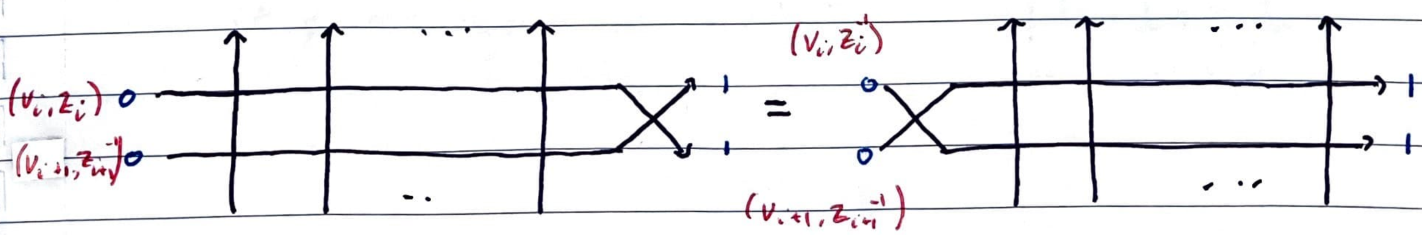
Proof.

(1) This is the standard YB argument, noting that



$= 1$

(2) Note the identity



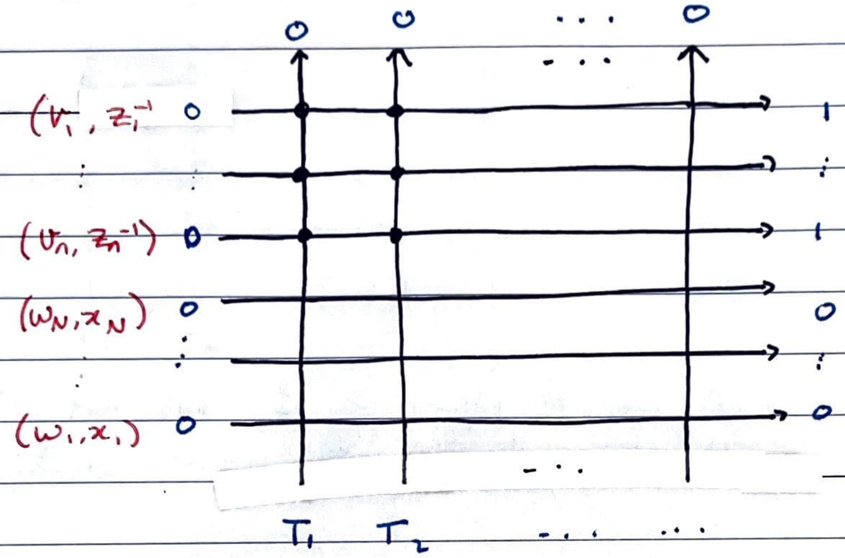
This leads to the identity

$$F_S(z_1,\dots,z_n; v_1,\dots,v_n)(v_i + z_{i+1}^{-1}) = F_S(\dots,z_{i+1},z_i,\dots;\dots,v_{i+1},v_i,\dots)(v_{i+1}+z_i^{-1})$$

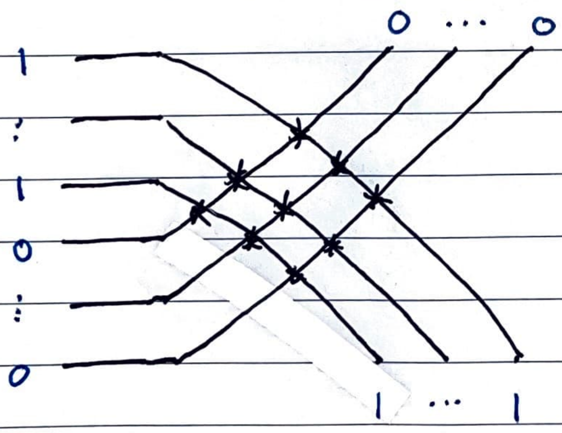
(3) Start from the identity

$$\sum_S G_{S,T}(x;w) F_S(z;v)$$

$= \sum_S$



To the RHS of the diagram, attach the cross

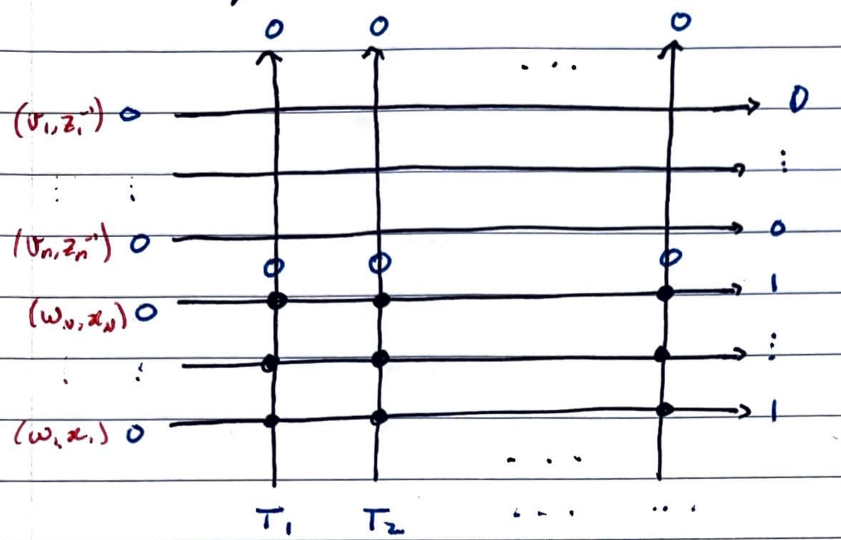


, which is completely frozen (claim)

The weight of this region is equal to $\prod_{i=1}^{N} \prod_{j=1}^{n} \frac{1-x_i z_j}{1+w_i z_j}$.

By YB, one takes the cross to the left, where it freezes off with weight 1.

What is left is :



where the sum over S degenerates to one term.
This picture is precisely $F_T(z;v)$ $\square$.