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23rd Sep 2025

Exactly Solvable Models
Lecture 26

Last time

(1) $$s_\lambda(x_1,\dots,x_n) = \frac{\det\limits_{1\le i,j\le n}\left(x_i^{\lambda_i - j + n}\right)}{\prod\limits_{1\le i<j\le n}(x_i - x_j)}$$ } Bethe Ansatz

(2) $$s_\lambda(x_1,\dots,x_n) = \det_{1\le i,j\le \ell(\lambda)}\left(h_{\lambda_i - i + j}(x)\right) = \det_{1\le i,j\le \ell(\lambda')}\left(e_{\lambda'_i - i + j}(x)\right)$$ } non-intersecting lattice paths + LGV (Lindström-Gessel-Viennot Lemma)

(3) $$s_\lambda(x_1,\dots,x_n) = \sum_{\emptyset=\lambda^{(0)}\prec\lambda^{(1)}\prec\cdots\prec\lambda^{(n)}=\lambda} \prod_{i=1}^{n} s_{\lambda^{(i)}/\lambda^{(i+1)}}(x_i)$$ } rows decomposition of a partition function

$$s_\lambda(x_1,\dots,x_n) = \sum_{T\in SSYT_n(\lambda)} x^T$$ } tableau formula

It turns out that the formulas of (3) are equivalent.

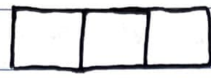
We sketch this in an example.
Consider $\lambda = (5\ 4\ 4\ 2\ 1)$ and $n=5$.
Define

$$T = \begin{matrix} 1 & 1 & 1 & 2 & 4 \\ 2 & 2 & 3 & 4 & \\ 3 & 3 & 4 & 5 & \\ 4 & 5 & & & \\ 5 & & & & \end{matrix}.$$

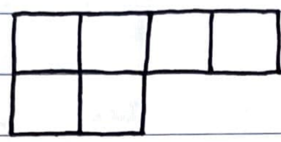
From here we wish to obtain an interlacing sequence of diagrams.

We do this in the following way:

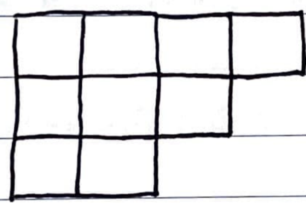
Define $\lambda^{(1)}$ = the diagram formed by the 1s

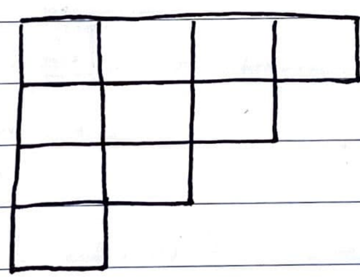
=  $= (3)$

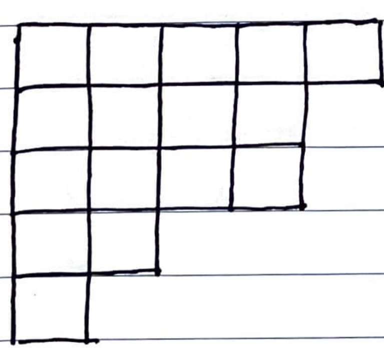
$\lambda^{(2)}$ = the diagram formed by the 1s and 2s

=  $= (4\ 2)$

$\lambda^{(3)}$ = the diagram formed by the 1s and 2s and 3s

=  $= (4\ 3\ 2)$

$\lambda^{(4)}$ =  $= (4\ 3\ 2\ 1)$

$\lambda^{(5)}$ =  $= (5\ 4\ 4\ 2\ 1)$

The claim is that $\emptyset \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \lambda^{(3)} \prec \lambda^{(4)} \prec \lambda^{(5)} = \lambda$ and this gives a bijection between tableaux and interlacing sequences.

Furthermore, the summands match. That is,

$$x_1^{|\lambda^{(1)}| - |\lambda^{(0)}|} \; x_2^{|\lambda^{(2)} - \lambda^{(1)}|} \cdots x_s^{|\lambda^{(s)}| - |\lambda^{(4)}|}$$

$$= \prod_{i=1}^{s} x_i^{\#\text{ of incidences of } i \text{ in } T}$$

§4. Cauchy Identities

Schur polynomials satisfy (infinite) summation identities that come in two types.

Theorem Fix $N \geq 1$ and a partition μ. Then, we have

(a) $$\sum_{\lambda} s_{\lambda/\mu}(x_1, \dots, x_N)\, s_\lambda(z_1, \dots, z_n) = \prod_{i=1}^{N} \prod_{j=1}^{n} \frac{1}{1 - x_i z_j}$$ (Cauchy Identity)

(This either holds as a formal power series, or as a convergent series assuming $|x_i z_j| < 1$ for all i, j.)

(b) $$\sum_{\lambda} s_{\lambda'/\mu'}(x_1, \dots, x_N)\, s_\lambda(z_1, \dots, z_n) = \prod_{i=1}^{N} \prod_{j=1}^{n} (1 + x_i z_j)\, s_\mu(z_1, \dots, z_n)$$

(dual Cauchy identity)

(This is actually a finite sum (support is finite). One finds that $\ell(\lambda) \leq n$ and $\lambda_1 \leq N + \mu_1$, so both length and size of the largest part are bounded)

Corollary Taking $N=1$, we recover the Pieri identities:

(a) $$\sum_{\lambda} \mathbb{1}_{\lambda \succ \mu} \, x^{|\lambda|-|\mu|} s_\lambda(z) = \left(\sum_{k=1}^{\infty} h_k(z) x^k \right) s_\mu(z).$$

Extracting powers of x, we get

$$\underline{s_\mu(z) \cdot h_k(z)} = \sum_{\substack{\lambda \\ |\lambda|-|\mu|=k \\ \lambda \succ \mu}} s_\lambda(z)$$

S

(horizontal strips)

(b) $$\sum_{\lambda} \mathbb{1}_{\lambda' \succ \mu'} \, x^{|\lambda|-|\mu|} s_\lambda(z) = \left(\sum_{k=1}^{\infty} e_k(z) x^k \right) s_\mu(z)$$

Extracting powers of x, we get

$$\underline{s_\mu(z) \cdot e_k(z)} = \sum_{\substack{\lambda \\ |\lambda|-|\mu|=k \\ \lambda' \succ \mu'}} s_\lambda(z)$$

S

(vertical strips)

Both (a) and (b) are special cases of the following problem:

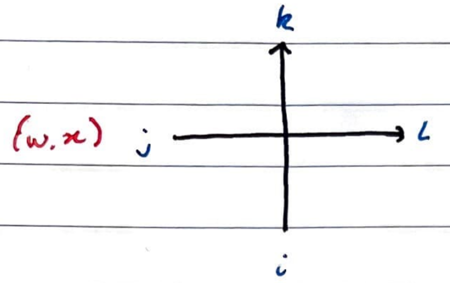
$$\underline{s_\mu(z)\, s_\nu(z)} = \sum_{\lambda} c_{\mu\nu}^{\lambda} \, s_\lambda(z)$$

Littlewood-Richardson coefficients

§.5. Another six-vertex model

We will recover s_λ from vertex models.
(Warning: there are many ways to do this)

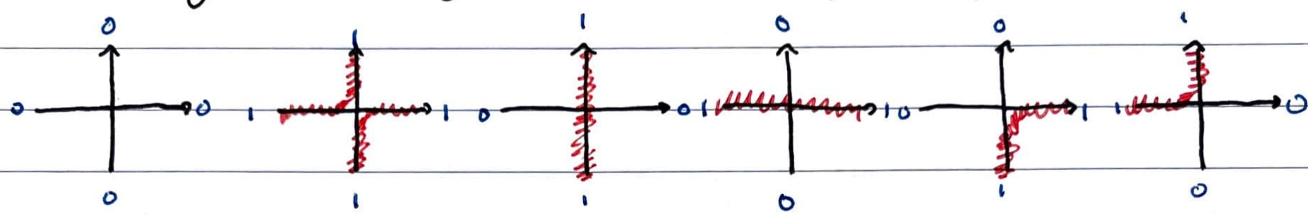
Our new six-vertex model has the generic form



$i, j, k, l \in \{0, 1\}$

(horizontal row has two spectral parameters)

The six weights that obey conservation are given by:



1	w	1	x	$x+w$	1

From these weights, we may construct an L matrix:

$$\underline{L_{bc}} = \begin{pmatrix} 1 & & & \\ & 1 & x+w & \\ & 1 & x & \\ & & & w \end{pmatrix} \in \mathrm{End}(V_b \otimes V_c).$$

Then, it satisfies

$$R_{ab} L_{ac}(v, z) L_{bc}(w, x) = L_{bc}(w, x) L_{ac}(v, z) R_{ab}.$$

where $$R_{ab} = \begin{pmatrix} w+z & & & \\ & w-v & v+z & \\ & w+x & z-x & \\ & & & v+x \end{pmatrix} \in \mathrm{End}(V_a \otimes V_b).$$

(all $6 = \binom{4}{2}$ pairs of variables appear!).