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Exactly Solvable Models
Lecture 25

4) Towards Algebraic Combinatorics

We study algebraic combinatorics via vertex models:

(1) Properties of Schur polynomials;
(2) Proof of properties via vertex models;
(3) Probability Measures → Schur measure (Okounkov-Reshetikin);
(4) Random tilings (lozenge tilings);
(5) Limit of (3) to the Gaussian Unitary Ensemble.

§1. Schur polynomials

These are a family of symmetric polynomials that are famous for (at least) two reasons:

(1) Characters of polynomial representations of $GL(n)$
(see Stanley Enumerative Combinatorics Vol. 2)

(2) Consider $Gr(k,n) = \{ \emptyset \subseteq V \subseteq W \mid \dim(V) = k,\ \dim(W) = n \}$.
One may decompose $Gr(k,n)$ to Schubert cells – the Schur polynomials are a basis for the cohomology ring of $Gr(k,n)$.

Definition Fix $n \in \mathbb{Z}_{\geq 1}$ and a partition $\lambda = (\lambda_1, \dots, \lambda_n)$. Define

$$s_\lambda(x_1, \dots, x_n) = \frac{\det_{1\leq i,j\leq n}\left(x_i^{\lambda_i - j + n}\right)}{\det_{1\leq i,j\leq n}\left(x_i^{n-j}\right)} = \frac{\det_{1\leq i,j\leq n}\left(x_i^{\lambda_i - j + n}\right)}{\prod_{1\leq i<j\leq n}(x_i - x_j)}.$$

If $\ell(\lambda) > n$, then define $s_\lambda = 0$.

Exercise Show that $s_\lambda(x_1, \ldots, x_n)$ is a symmetric polynomial.

The Schur polynomials satisfy the stability property:

$$s_\lambda(x_1, \ldots, x_n, x_{n+1})\big|_{x_{n+1}=0} = s_\lambda(x_1, \ldots, x_n).$$

This allows one to extend the definition of a Schur function to alphabets of arbitrary (possibly infinite) (See Macdonald)

§2. Jacobi-Trudi formulae

Definition The complete symmetric functions $h_k(x)$ are given by

$$h_k(x) = \mathrm{Coeff}_{z^k}\left(\prod_{i=1}^{n} \frac{1}{1-x_i z}\right) \qquad (S\;)$$

The elementary symmetric functions $e_k(x)$ are given by

$$e_k(x) = \mathrm{Coeff}_{z^k}\left(\prod_{i=1}^{n} 1+x_i z\right) \qquad (S\;)$$

Theorem The Schur functions are also given by

$$s_\lambda(x_1, \ldots, x_n) = \det_{1 \le i,j \le \ell(\lambda)} \left(h_{\lambda_i - i + j}(x)\right)$$

non-trivial! Schur involution.

$$= \det_{1 \le i,j \le \ell(\lambda')} \left(e_{\lambda'_i - i + j}(x)\right)$$

§3. Branching rule and tableaux

Fact: the Schur polynomials are a $\mathbb{Z}$-basis for the ring of symmetric polynomials in n-variables $\mathbb{Z}[x_1, \dots, x_n]^{S_n}$.

One can consider taking $s_\lambda(x_1, \dots, x_n)$ and regarding x_n as a 'constant'. Then, one expands $s_\lambda(x_1, \dots, x_n)$ over the Schur basis in $n-1$ variables.

We obtain

$$s_\lambda(x_1, \dots, x_n) = \sum_\mu s_\mu(x_1, \dots, x_{n-1})\, s_{\lambda/\mu}(x_n)$$

for certain coefficients $s_{\lambda/\mu}(x_n)$ called the branching coefficients.

Theorem

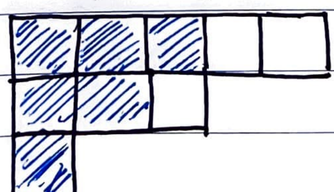
$$s_{\lambda/\mu}(z) = \begin{cases} z^{|\lambda| - |\mu|}, & \lambda \succ \mu \\ 0, & \text{otherwise}. \end{cases}$$

where $|\lambda| = \sum_{i=1}^{\ell(\lambda)} \lambda_i$ and

$$\lambda \succ \mu \quad (\lambda \text{ 'interlaces' } \mu) \quad \text{if} \quad \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots$$

(this is equivalent to saying $\lambda - \mu$ forms horizontal strips)

Example Consider the interlacing string $(5\,3\,3\,2\,1\,1\,1)$.
We read $\lambda = (5\;3\;1\;1)$ and $\mu = (3\;2\;1)$.
Notice

$$\lambda/\mu =$$  .

It is also possible to branch several variables.

$$s_\lambda(x_1,\dots,x_n) = \sum_\mu s_\mu(x_1,\dots,x_m)\, s_{\lambda/\mu}(x_{m+1},\dots,x_n)$$

skew Schur polynomial.

By iteration of (b), we obtain:

$$s_\lambda(x_1,\dots,x_n) = \sum_{\emptyset=\lambda^{(0)}\prec\lambda^{(1)}\prec\dots\prec\lambda^{(n)}=\lambda} \prod_{i=1}^{n} s_{\lambda^{(i)}/\lambda^{(i-1)}}(x_i).$$

This formula is equivalent to the 'semi-standard Young tableau' formula:

$$s_\lambda(x_1,\dots,x_n) = \sum_{T\in SSYT(\lambda)} x^T.$$

Here, T is a filling of the Young diagram of λ by numbers $\{1,\dots,n\}$ such that:

(1) Numbers weakly increase along rows

(2) Numbers strictly increase along columns

Then, $$x^T = \prod_{i=1}^{n} x_i^{\#(\text{instances of } i \text{ in } T)}.$$