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Exactly Solvable Models
Lecture 24

Recap:

- We set all vertical parameters y_i to y (i.e make them the same).

- Our old functions $G^{old}_{S/T}$ and F^{old}_{S} were indexed by strings that were infinite in one direction e.g. $S = (S_1, S_2, \dots)$.

- We wish to define $G^{new}_{S/T}$ and F^{new}_{S} where the strings are infinite in two directions e.g. $S = (\dots, S_{-1}, S_0, S_1, \dots)$ with the condition $S_k = 0$ for all $|k|$ sufficiently large.

To do this, we first define

$$G^{new}_{(0,\dots,0:0:S_1,S_2,\dots)/(0,\dots,0:0:T_1,T_2,\dots)} = G^{old}_{(S_1,\dots)/(T_1,\dots)}$$

Then, for arbitrary S, T

$$G^{new}_{S/T} = G^{new}_{\Delta^m S/\Delta^m T}, \qquad \text{for sufficiently large } m$$

Also, we define

$$F^{new}_{(0,\dots,0:0:S_1,S_2,\dots)} = F^{old}_{(S_1,S_2,\dots)}$$

Then, all others are given by

$$F^{new}_{S}(z|y) = \prod_{i=1}^{n} \left(\frac{1 - z_i/y}{1 - q z_i/y} \right)^{M} F^{new}_{\Delta^m S}(z|y)$$

Note that

$$F_S^{new}(z^{-1}|qy^{-1}) = q^{-nM} \prod_{i=1}^{n} \left(\frac{1 - qz_i/y}{1 - z_i/y} \right)^M F_{\Delta^m S}^{new}(z^{-1}|qy^{-1}).$$

We can now write down an integral formula for $G_{S/T}^{new}$:

$$G_{S/T}^{new}(x_1, \dots, x_N | y)$$

$$= G_{\Delta^m S/\Delta^m T}^{new}(x_1, \dots, x_N | y)$$

$$= \frac{1}{C_{\Delta^m S}(q)} \oint \frac{dz_1}{2\pi i z_1} \cdots \oint \frac{dz_n}{2\pi i z_n} \prod_{1 \le i \ne j \le n} \frac{z_i - z_j}{z_i - q z_j} \prod_{i=1}^{N} \prod_{i=1}^{n} \frac{x_i - q z_j}{x_i - z_j}$$

$$F_{\Delta^m S}^{new}(z^{-1}|qy^{-1}) \, F_{\Delta^m T}^{new}(z|y)$$

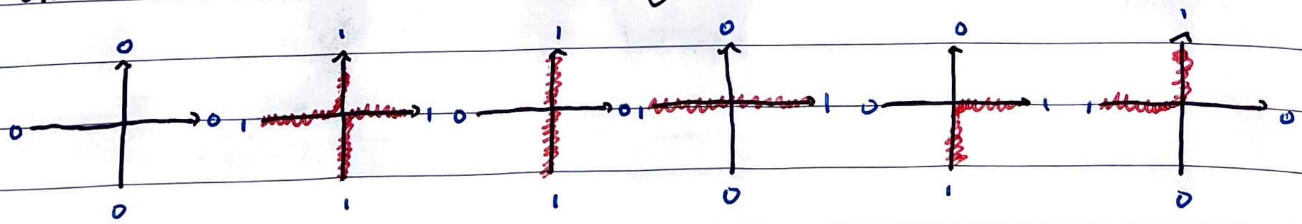
$$= \frac{q^{nM}}{C_{\Delta^m S}(q)} \oint \frac{dz_1}{2\pi i z_1} \cdots \oint \frac{dz_n}{2\pi i z_n} \prod_{1 \le i \ne j \le n} \frac{z_i - z_j}{z_i - q z_j} \prod_{i=1}^{N} \prod_{i=1}^{n} \frac{x_i - q z_j}{x_i - z_j}$$

$$F_S^{new}(z^{-1}|qy^{-1}) \cdot F_T^{new}(z|y)$$

Henceforth, we will drop the 'new' superscript.

Now, we consider a limit of the stochastic 6V model where we choose $y = 1$ and $x_i = 1 - (1-q)\varepsilon$.

Under this choice, the vertex weights become

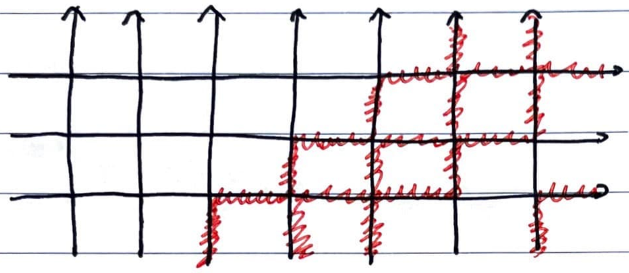


1 $\quad$ 1 $\quad$ $\frac{q\varepsilon}{1+q\varepsilon}$ $\quad$ $\frac{\varepsilon}{1+q\varepsilon}$ $\quad$ $\frac{1}{1+q\varepsilon}$ $\quad$ $\frac{1-(1-q)\varepsilon}{1+q\varepsilon}$

Remark To $O(\varepsilon)$ the vertex weight are

$$1, \quad 1, \quad q\varepsilon, \quad \varepsilon, \quad 1-q\varepsilon, \quad 1-\varepsilon$$

Studying the function $G_{S/T}$ in this limit, we see that paths tend to follow a zig-zag motion:



This motivates the following:

Theorem (Aggarwal)

$$\lim_{N\to\infty} G_{S/\Delta^{Nt}T}\Big(\underbrace{1-\frac{(1-q)}{N}, \dots, 1-\frac{(1-q)}{N}}_{Nt \text{ variables}}\Big|\, 1\Big) = \mathbb{P}_t^{ASEP}(T\to S)$$

We see what this does to the integral:

$$G_{S/\Delta^{Nt}T}\left(1-\frac{(1-q)}{N}, \dots, 1-\frac{(1-q)}{N}\,\Big|\, 1\right)$$

$$= (\text{const that depends on } S \text{ and } q) \cdot \oint \frac{dz_1}{2\pi i z_1} \dots \oint \frac{dz_n}{2\pi i z_n} \prod_{1\le i\ne j\le n} \frac{z_i - z_j}{z_i - qz_j}$$

$$\prod_{j=1}^{n}\left(\frac{qz_j - 1 + \frac{(1-q)}{N}}{z_j - 1 + \frac{(1-q)}{N}}\right)^{Nt} F_S(z^{-1}|q) \prod_{j=1}^{n}\left(\frac{1-z_j}{1-qz_j}\right)^{Nt} F_T(z|1)$$

Grouping the underlined terms, they yield

$$\prod_{j=1}^{n}\left(1+\frac{(1-q)^2 z_j}{(1-z_j)(1-qz_j)N}+O(N^{-2})\right)^{Nt}.$$

This will converge to an exponential as $N\to\infty$.

We conclude, from the previous theorem,

$$\underline{\mathbb{P}_t^{ASEP}(T\to S)}=(\text{const.})\oint\frac{dz_1}{2\pi i z_1}\cdots\oint\frac{dz_n}{2\pi i z_n}\prod_{1\le i<j\le n}\frac{z_i-z_j}{z_i-qz_j}$$

$$\prod_{j=1}^{n}\exp\left(\frac{(1-q)^2 z_j}{(1-z_j)(1-qz_j)}\right)F_S(z^{-1}|q)\,F_T(z|1).$$

c.f. Tracy-Widom 'Bethe Ansatz Formula'.