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Exactly Solvable Models
Lecture 23

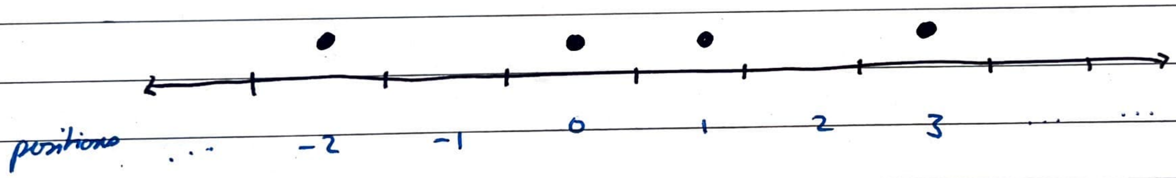
Last time:

$$G_{S/T}(x_1,\dots,x_N) = \frac{1}{C_S(q)} \oint \frac{dz_1}{2\pi i z_1} \cdots \oint \frac{dz_N}{2\pi i z_N} \prod_{1\le i<j\le N} \frac{z_i - z_j}{z_i - q z_j} \prod_{i=1}^{N}\prod_{j=1}^{N} \frac{x_i - q z_i}{x_i - z_j} \cdot F_S(z^{-1} \mid q y^{-1}) \cdot F_T(z \mid y)$$

§9. Asymmetric simple exclusion process (ASEP)

This is a continuous time Markov process that lives on the integer lattice.

The state space of the process is the set of subsets of $\mathbb{Z}$.



We will label the states by strings S.

We consider finitely many particles, so for each of our strings S, there exists M such that if $|i| \geq M$ then $S_i = 0$

The ASEP is defined by the following rules:

(1) Each particle in the system has a left and right alarm clock associated to it.

(2) The left-clocks are ringing with rate q (less than 1).
The right-clocks are ringing with rate 1.
All ringing times are independent.

More precisely, this means that the waiting time for a left-clock to ring is a random variable τ_L which has distribution

$$\mathbb{P}(\tau_L \leq s) = q\int_0^s e^{-qt}\,dt = 1 - e^{-qs}.$$

Similarly, for the right-clock we have

$$\mathbb{P}(\tau_R \leq s) = \int_0^s e^{-t}\,dt = 1 - e^{-s}$$

(3) When a left- or right-clock rings, the associated particle (if able to) performs a left-or right- jump.

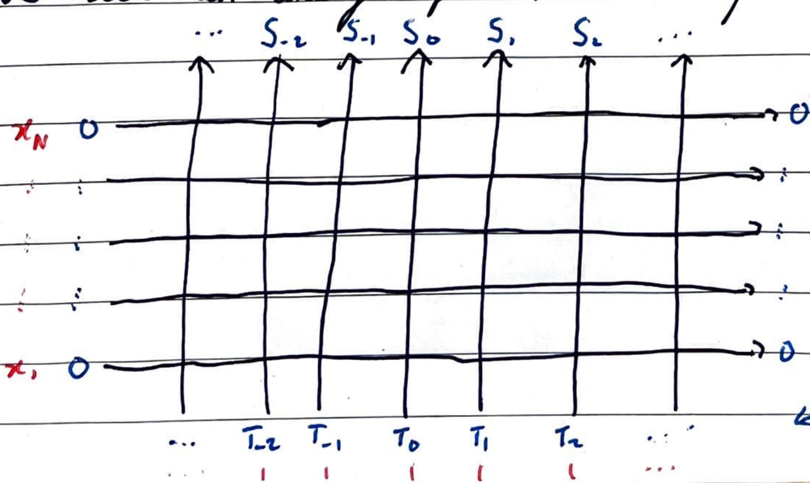
(4) All other particles during a jump remain stationary. After the jump we reset all the clocks and run this procedure again.

The quantity of interest is

$P_t(T \to S)$ = the probability of being in state S at time t given the system was in state T at time 0.

§10. Limit from six-vertex to ASEP

We seek an analogue of $G_{S/T}$ on the full integer line:



$= G_{S/T}(x_1, \dots, x_N)$

y spectral parameters are now all 1.

We could also define $G_{S/T}$ by translation invariance:

$$G_{S/T} = G_{\Delta S/\Delta T}$$

where $(\Delta S)_i = S_{i-1}$ for each $i \in \mathbb{Z}$.

Next, the function F_S is currently defined for

$$S = (S_1, S_2, \ldots).$$

However, note the following:

<u>Proposition 3.11</u> Fix a string $S = (S_1, S_2, \ldots)$ and define

$$\Delta S = (0, S_1, S_2, \ldots)$$

Then,

$$F_{\Delta S}(z_1, \ldots, z_n \mid 1) = \prod_{i=1}^{n} \frac{1 - q z_i}{1 - z_i} F_S(z_1, \ldots, z_n \mid 1)$$

This allows us to write

$$F_S(z_1, \ldots, z_n) = \prod_{i=1}^{n} \left(\frac{1 - z_i}{1 - q z_i} \right)^M F_{\Delta^M S}(z_1, \ldots, z_n)$$

where M is sufficiently large such that $(\Delta^M S)_i = 0$ for all $i < 0$.