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Exactly Solvable Models
Lecture 22

§7. Orthogonality

Recall that given an $n \times n$ matrix M with left- and right-eigenvectors $|V_k\rangle$ and $\langle W_k|$ and eigenvalues λ_k, one has

(1) $I = \sum_k |V_k\rangle\langle W_k|$ (decomposition of the identity)

(2) $M = \sum_k \lambda_k |V_k\rangle\langle W_k|$ (spectral decomposition)

We seek an analogue of (1) in the case of our eigenvectors:

$$|\{z_1,\dots,z_n\}\rangle = |F(z_1,\dots,z_n)\rangle$$

$$= \sum_S F_S(z_1,\dots,z_n)|S\rangle$$

Theorem 3.10 Fix two strings S,T such that $|S|=|T|=n\geq 1$. Then, there exist integration contours $\mathcal{C}_1,\dots,\mathcal{C}_n$ and a constant $C_S(q)$ such that

$$\oint_{\mathcal{C}_1} \frac{dz_1}{2\pi i z_1} \cdots \oint_{\mathcal{C}_n} \frac{dz_n}{2\pi i z_n} \prod_{1\leq i\neq j\leq n} \frac{z_i - z_j}{z_i - q z_j} \, F_S(z_1,\dots,z_n|y)$$

$$F_T(z_1^{-1},\dots,z_n^{-1}|qy^{-1})$$

$$= \delta_{S,T}\, C_S(q).$$

Remarks

(1) We could also write

$$\oint_{\mathcal{C}_1} \frac{dz_1}{2\pi i z_1} \cdots \oint_{\mathcal{C}_n} \frac{dz_n}{2\pi i z_n} \prod_{1\le i\ne j\le n} \frac{z_i - z_j}{z_i - q z_j} \, |F(z_1,\dots,z_n)|y\rangle \langle F(z_1^{-1},\dots,z_n^{-1}|q y^{-1})|$$

$= \mathbb{D}_n$ (a diagonal matrix)

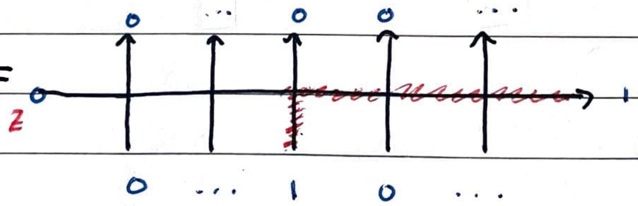
Proof (General n proof is in Assignment 2)

We focus on $n=1$.
So,

s^{th} position $\qquad\qquad$ t^{th} position

$$S = (0,\dots,0,1,0,\dots,0), \qquad T = (0,\dots,0,1,0,\dots,0)$$

We explicitly compute F_S and F_T.

$F_S(z|y) =$ 

$$= \left(\prod_{i=1}^{s-1} \frac{1 - qz/y_i}{1 - z/y_i}\right)\left(\frac{1-q}{1 - z/y_s}\right)$$

Similarly,

$$F_T(z^{-1}|q y^{-1}) = \left(\prod_{i=1}^{t-1} \frac{1 - qz^{-1}/qy_i^{-1}}{1 - z^{-1}/qy_i^{-1}}\right)\left(\frac{1-q}{1 - z^{-1}/qy_t^{-1}}\right)$$

$$= \left(\prod_{i=1}^{t-1} \frac{1 - y_i/z}{1 - y_i/qz}\right)\left(\frac{(1-q)}{1 - y_t/qz}\right)$$

$$= \left(\prod_{i=1}^{t-1} \frac{qz - qy_i}{qz - y_i} \right) \left(\frac{(1-q)qz}{qz - y_t} \right)$$

$$= q^t \left(\prod_{i=1}^{t-1} \frac{z - y_i}{qz - y_i} \right) \left(\frac{(1-q)z}{qz - y_t} \right)$$

$$= q^t \left(\prod_{i=1}^{t-1} \frac{z/y_i - 1}{qz/y_i - 1} \right) \left(\frac{(1-q)z/y_t}{qz/y_t - 1} \right)$$

$$= q^t \left(\prod_{i=1}^{t-1} \frac{1 - z/y_i}{1 - qz/y_i} \right) \left(\frac{(q-1)z/y_t}{1 - qz/y_t} \right)$$

So, because of partial cancellation, we get

$F_S(z|y)F_T(z^{-1}|qy^{-1})$

$$= \begin{cases} \text{(a)} \quad -\prod_{i=s+1}^{t-1} \left(\frac{1 - z/y_i}{1 - qz/y_i} \right) \left(\frac{q^t (1-q)^2 z/y_t}{(1 - qz/y_t)(1 - qz/y_s)} \right) & , \; s < t \\ \text{(b)} \quad -\prod_{i=t+1}^{s-1} \left(\frac{1 - qz/y_i}{1 - z/y_i} \right) \left(\frac{q^t (1-q)^2 z/y_t}{(1 - z/y_t)(1 - z/y_s)} \right) & , \; s > t \\ \text{(c)} \quad -\frac{q^t (1-q)^2 z/y_t}{(1 - z/y_t)(1 - qz/y_t)} & , \; s = t \end{cases}$$

In cases (a) and (b), the pole structure is of one type

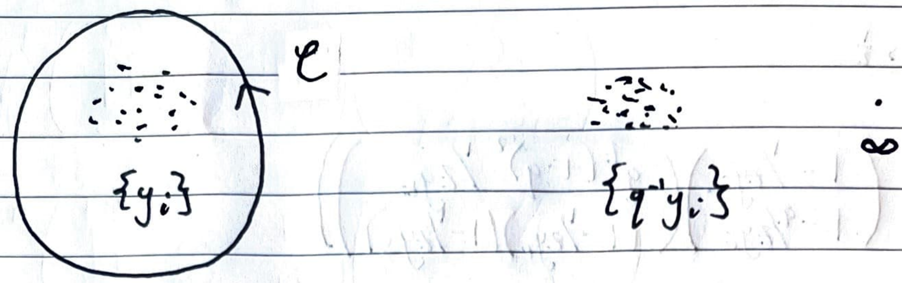
↳ In (a), we have poles at $1 - qz/y_i = 0 \Leftrightarrow z = q^{-1}y_i$, $i \in \{s, \dots, t\}$.

↳ In (b), we have poles at $1 - z/y_i = 0 \Leftrightarrow z = y_i$, $i \in \{t, \dots, s\}$

In case (c), both types of poles occur.

↳ We have poles at $z = y_t$ and $z = q^{-1}y_t$.

We shall choose $\mathcal{C}_1 = \mathcal{C}$ as follows:
(note that there is an argument that shows that this is sufficiently general)



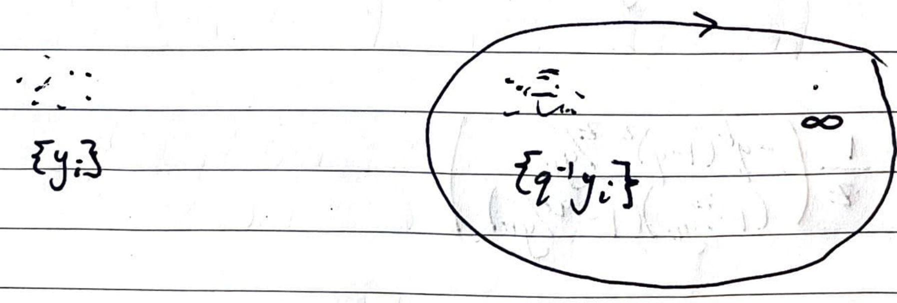
We now use the famous result that

$$\oint \frac{dw}{2\pi i w} w^k w^{-l} = \delta_{k,l}$$

In case (a), all poles are of type $z = q^{-1}y_i$ so

$$\oint_{\mathcal{C}} \frac{dz}{2\pi i z} (a) = 0$$

In case (b), we consider instead the contour



So, it suffices to check the residue at ∞.
We have:

$$\oint_{\mathcal{C}} \frac{dz}{2\pi i z} (b) = -\mathrm{Res}\left(\frac{1}{z}\cdot(b), \infty\right)$$

$$= \mathrm{Res}\left(\frac{1}{z^2}\left(\frac{1}{z}(b)\right)\Big|_{z\to\frac{1}{z}}, 0\right)$$

Yet,

$$\frac{1}{z^2}\left(\frac{1}{z}\,(b)\right)_{z\to\frac{1}{z}}$$

$$= -\frac{1}{z^2}\cdot z\left(\prod_{i=t+1}^{s-1}\left(\frac{1-1/zy_i}{1-q/zy_i}\right)\left(\frac{q^t(1-q)^2\,1/zy_t}{(1-1/zy_t)(1-1/zy_s)}\right)\right)$$

$$=(-1)\left(\prod_{i=t+1}^{s-1}\frac{z-1/y_i}{z-q/y_i}\right)\left(\frac{q^t(1-q)^2\,1/y_t}{(z-1/y_t)(z-1/y_s)}\right)$$

Since this is a rational function of degree -2 with no poles at $z=0$, it follows that

$$\oint_{\mathcal{C}}\frac{dz}{2\pi i z}\,(b)=0$$

Finally, in case (c), we get

$$\oint_{\mathcal{C}}\frac{dz}{2\pi i z}\,(c)=\operatorname{Res}\left(\tfrac{1}{z}\,(c),\,y_t\right)$$

$$=\lim_{z\to y_t}(z-y_t)\,\frac{1}{z}\left(\frac{-q^t(1-q)^2\,z/y_t}{(1-z/y_t)(1-qz/y_t)}\right)$$

$$=\lim_{z\to y_t}\frac{z-y_t}{1-z/y_t}\cdot\frac{1}{y_t}\cdot\frac{-q^t(1-q)^2\cdot 1}{1-q}$$

$$=\frac{(-y_t)}{y_t}\,\left(-q^t(1-q)\right)$$

$$=q^t(1-q)$$

It follows that

$$\oint_{\mathcal{C}} \frac{dz}{2\pi i z} F_S(z|y) \cdot F_T(z^{-1}|qy^{-1})$$

$$= \begin{cases} 0, & s < t \\ 0, & t < s \\ q^s(1-q), & s = t \end{cases}$$

□

§8. Integral Formula for $G_{S/T}$

Recall the Cauchy identity

$$\sum_{u} G_{u/T}(x|y) F_u(z|y) = \prod(x;z) F_T(z|y).$$

We proceed by : (1) Multiplying both sides by $F_S(z^{-1}|qy^{-1})$.
(2) Integrating as in the orthogonality theorem.

Assuming uniform convergence of the Cauchy sum, we get,

$$\sum_{u} \oint_{\mathcal{C}_1} \frac{dz_1}{2\pi i z_1} \cdots \oint_{\mathcal{C}_n} \frac{dz_n}{2\pi i z_n} \prod_{1\le i\ne j\le n} \frac{z_i - z_j}{z_i - qz_j} F_S(z^{-1}|qy^{-1}) G_{u/T}(x|y) F_u(z|y)$$

(by orthogonality)

$$= C_S(q)\, G_{S/T}(x|y)$$

(by equality with RHS)

$$= \oint_{\mathcal{C}_1} \frac{dz_1}{2\pi i z_1} \cdots \oint_{\mathcal{C}_n} \frac{dz_n}{2\pi i z_n} \prod_{1\le i\ne j\le n} \frac{z_i - z_j}{z_i - qz_j} \cdot \prod(x;z) F_S(z^{-1}|qy^{-1}) F_T(z|y)$$

This is the <u>spectral decomposition</u> of $G_{S/T}$.