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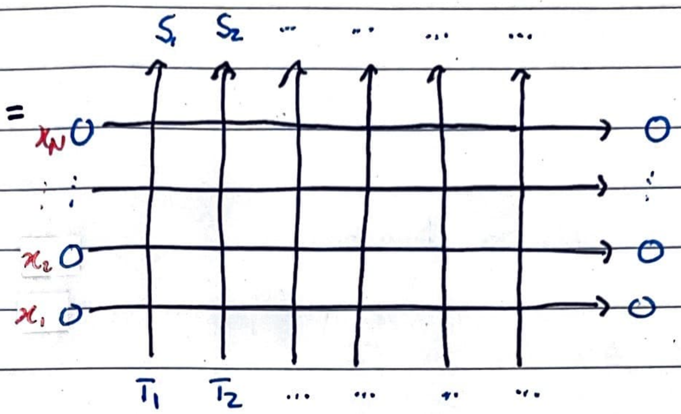
9th Sep 2025

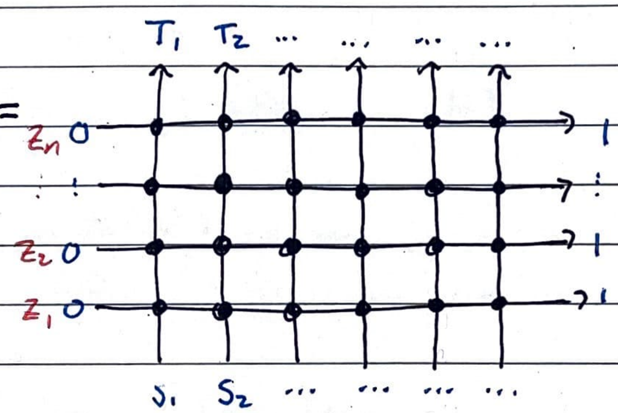
Exactly Solvable Models
Lecture 20

Recall

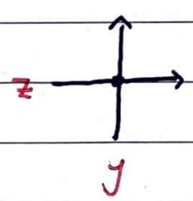
$G_{S/T}(x_1, \dots, x_N \mid y) =$



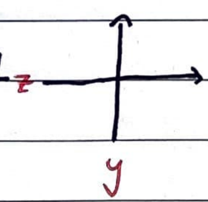
$F_S(z_1, \dots, z_n \mid y) =$



where



$= \dfrac{1 - qz/y}{1 - z/y}$



Theorem Fix a string T such that $|T| = \sum_{k=1}^{\infty} T_k = n$.
Then,

$$\sum_S G_{S/T}(x_1, \dots, x_N) \cdot F_S(z_1, \dots, z_n)$$

$$= \prod_{i=1}^{N} \prod_{j=1}^{n} \frac{1 - qz_j/x_i}{1 - z_j/x_i} \quad F_T(z_1, \dots, z_n),$$

provided that $\left| \dfrac{1 - x_i/y_k}{1 - q x_i/y_k} \cdot \dfrac{1 - q z_j y_k}{1 - z_j y_k} \right| < p < 1$

for all i, j, k.

<u>Remarks</u>

(1) This can be viewed as the components of the vector equation:

$$G(x_1, \dots, x_N) \,|\, F(z_1, \dots, z_n)\rangle\rangle = \prod_{i=1}^{N} \prod_{j=1}^{n} \frac{1 - q z_j / x_i}{1 - z_j / x_i} \,|F(z_1, \dots, z_n)\rangle\rangle$$

where $|F(z_1, \dots, z_n)\rangle\rangle = \sum_S F_S(z_1, \dots, z_n) \,|S\rangle$

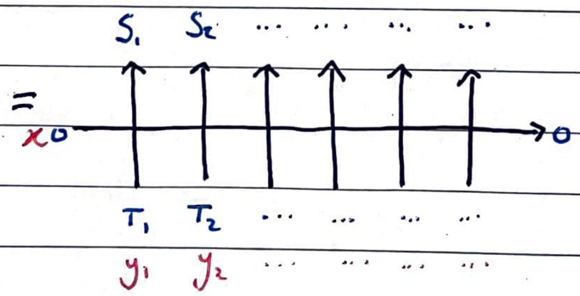
(2) This is actually related to performing algebraic Bethe ansatz in semi-infinite volume (no Bethe equations!)

(3) Summation identities of this type are called 'Cauchy Identities'. See part 4 of this course.

<u>Proof</u>. Define semi-infinite analogue of the monodromy matrix operations.

For finite strings S, T, construct the 'one-row' partition functions

$G_{S/T}(x) =$

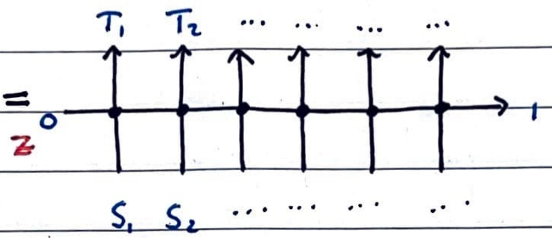


Arranging these into a matrix, we construct

$$A(x) = \left(G_{T/S}(x) \right)_{S,T}$$

Similarly, for finite strings S, T, construct the partition function

$$F_{S/T}(z) =$$



Again, define

(ROW INDEX: BOTTOM
COL INDEX: TOP)

$$B(x) = \left(F_{S/T}(z) \right)_{S,T}$$

We prove the following:

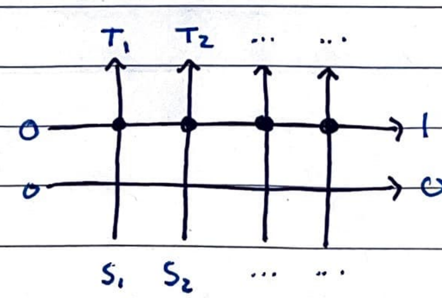
Lemma The operators satisfy:

$$A(x)B(z) = \frac{1 - qz/x}{1 - z/x} B(z)A(x)$$

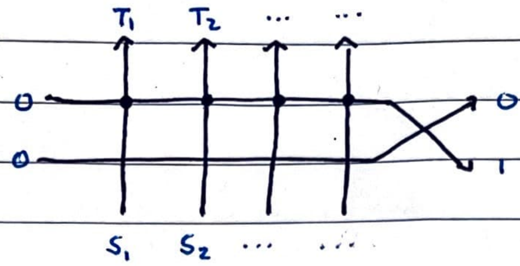
provided that $\left| \dfrac{1 - x/y_n}{1 - qx/y_n} \cdot \dfrac{1 - qz/y_n}{1 - z/y_n} \right| < p < 1$

Proof: We consider

$$\langle S | A(x) B(z) | T \rangle =$$

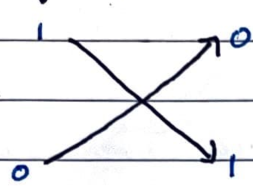


(claim)

$=$ 

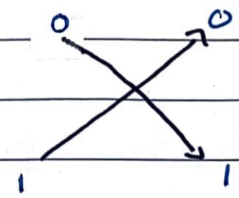
$$\cdot \left(\frac{1 - qz/x}{1 - z/x} \right)$$

The justification is:



(1)

$$= \frac{1 - z/x}{1 - qz/x}$$



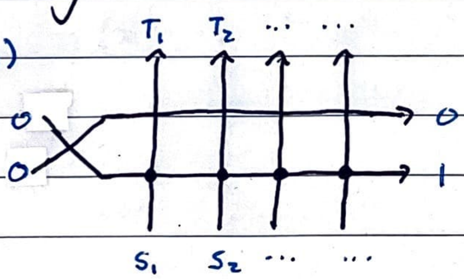
(2)

$$\frac{1-q}{1-qz/x}$$

One may show that the convergene conditions renders any configuration where (2) appears will have weight equal to 0.

Continuing:

(YB)

$=$ 

$$\cdot \left(\frac{1 - qz/x}{1 - z/x} \right)$$

(freezing)

$$= \left(\frac{1 - qz/x}{1 - z/x} \right)$$

$$= \langle S | B(z) A(x) | T \rangle$$

So,

$$\sum_S G_{S/T}(x_1,\dots,x_N)\,F_S(z_1,\dots,z_n)$$

$$= \sum_S \langle T|A(x_1)\cdots A(x_N)|S\rangle\langle S|B(z_1)\cdots B(z_n)|\emptyset\rangle$$

$$= \langle T|A(x_1)\cdots A(x_N)B(z_1)\cdots B(z_n)|\emptyset\rangle$$

$$= \langle T|B(z_1)\cdots B(z_n)A(x_1)\cdots A(x_N)|\emptyset\rangle\cdot\Pi(x;y)$$

$$= \langle T|B(z_1)\cdots B(z_n)|\emptyset\rangle\cdot\Pi(x;y)$$

$$= F_T(z_1,\dots,z_n)\cdot\Pi(x;y)$$

where $\Pi(x;y) = \prod_{i=1}^{N}\prod_{j=1}^{n}\frac{1-qz_j/x_i}{1-z_j/x_i}$ and we have used

the fact that $A(x)|\emptyset\rangle = |\emptyset\rangle$.