Gypsy Akhyar
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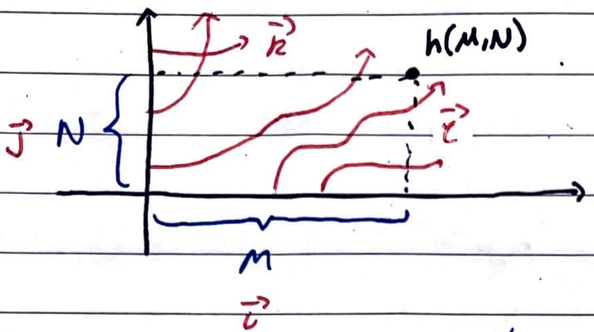
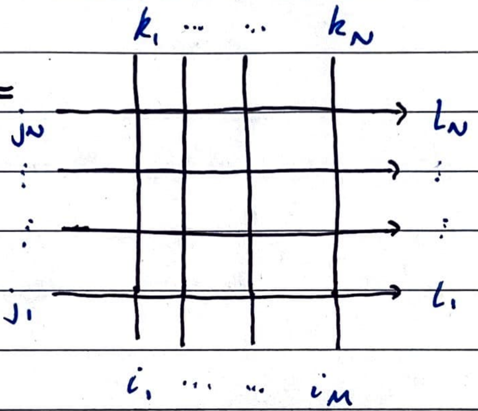
Exactly Solvable Models
Lecture 19

Last time:



$Z(\vec{i},\vec{j},\vec{k},\vec{\ell}\,) =$



$$= \mathbb{P}\big((\vec{i},\vec{j}) \to (\vec{k},\vec{\ell}\,)\big)$$

Remarks: (1) $Z = 0$ unless $|\vec{i}| + |\vec{j}| = |\vec{k}| + |\vec{\ell}|$

(2) If $\vec{i},\vec{j}$ are fixed, then $\displaystyle\sum_{\vec{k},\vec{\ell}} Z(\vec{i},\vec{j},\vec{k},\vec{\ell}\,) = 1.$

(3) Distribution of the height function

$$\mathbb{P}(h(M,N) = p) = \sum_{\vec{k},\vec{\ell}} \mathbb{1}_{|\vec{\ell}| = p}\, Z(\vec{i},\vec{j},\vec{k},\vec{\ell}\,)$$

§4. The function $G_{S/T}$

We define a distribution function as a special case of Z:

(1) Fix finite N and take $\vec{j} = \vec{0}$.

(2) Take $M \to \infty$ such that $|\vec{i}\,|$ is bounded.

Definition Fix two 'strings' S, T

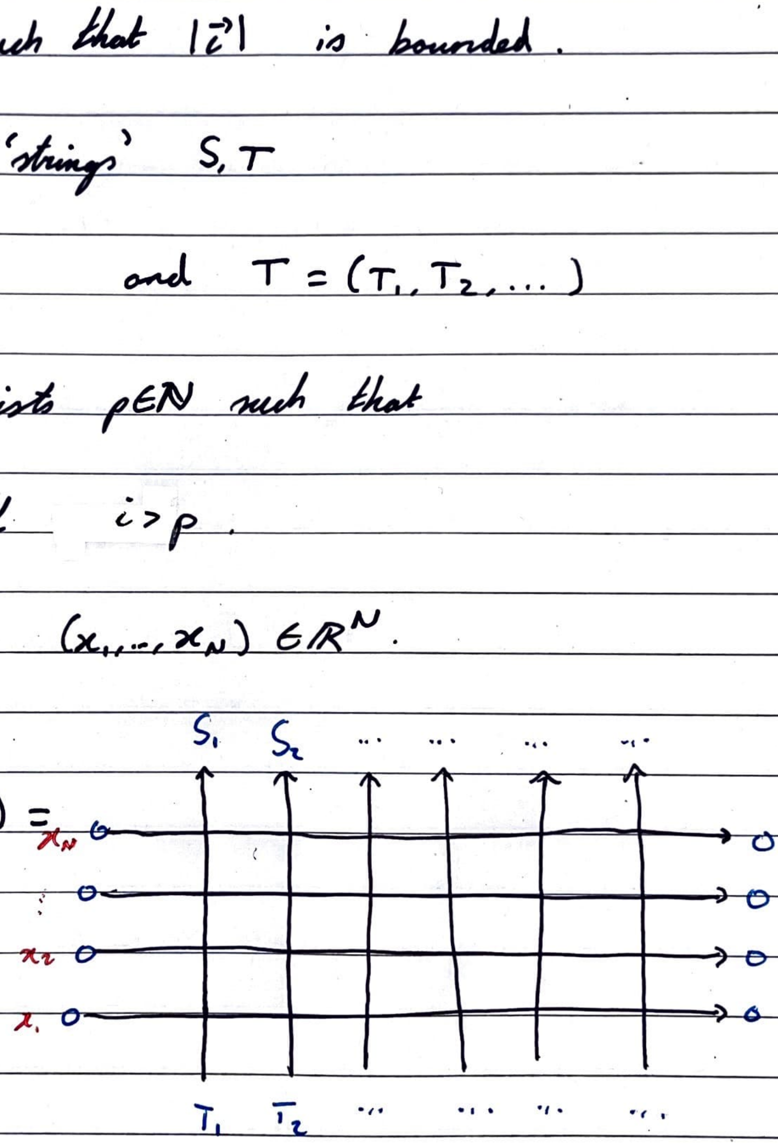
$$S = (S_1, S_2, \dots) \quad \text{and} \quad T = (T_1, T_2, \dots)$$

such that there exists $p \in \mathbb{N}$ such that

$$T_i = S_i = 0, \quad \text{if} \quad i > p.$$

Fix an alphabet $(x_1, \dots, x_N) \in \mathbb{R}^N$.
Define

$$G_{S/T}(x_1, \dots, x_N) =$$



(Implicitly, in $G_{S/T}$, we have vertical spectral parameters $y_1, y_2, \dots$. Even though we don't record them, they are still there.)

We shall derive an integral formula for $G_{S/T}$ over the coming lectures.

Remarks: (1) $\sum_{S} G_{S/T}(x_1, \ldots, x_N) = 1$

(2) $G_{S/T}$ is a symmetric function in $x_1, \ldots, x_N$. (This follows from the Yang-Baxter equation / commutativity of the A operators).

§5. Motivational Example

Let M be an $n \times n$ diagonalisable matrix with right and left eigenvectors $|v_k\rangle$ and $\langle w_l|$ for $k, l \in [1, n]$.

Suppose that $\{|v_k\rangle\}$ and $\{\langle w_l|\}$ are linearly independent and that their eigenvalues $\lambda_1, \ldots, \lambda_n$ are pairwise distinct.

Then, for $k \neq l$, we compute

$$\langle w_l | M | v_k \rangle \overset{M \text{ acting on } v_k}{=} \langle w_l | \cdot \lambda_k | v_k \rangle$$
$$= \lambda_k \cdot \langle w_l | v_k \rangle$$

and

$$\langle w_l | M | v_k \rangle \overset{M \text{ acting on } w_l}{=} \lambda_l \cdot \langle w_l | v_k \rangle .$$

So,

$$(\lambda_k - \lambda_l) \langle w_l | v_k \rangle = 0$$

and since $\lambda_k \neq \lambda_l$, then

$$\langle w_l | v_k \rangle = 0$$

Further, we can normalise such that $\langle w_k | v_k \rangle = 1$.

We now obtain the 'decomposition of the identity':

$$I = \sum_{k=1}^{n} |v_k\rangle\langle w_k| .$$

Notice that we may write an arbitrary $|v\rangle$ as

$$|v\rangle = \sum_{k=1}^{n} c_k |v_k\rangle .$$

Indeed, we obtain:

$$\left(\sum_{k=1}^{n} |v_k\rangle\langle w_k|\right)|v\rangle = \sum_{k,l=1}^{n} c_l |v_k\rangle\langle w_k|v_l\rangle$$

$$= \sum_{k=1}^{n} c_k |v_k\rangle \quad \text{(orthonormality)}$$

$$= |v\rangle .$$

We also have

$$M = MI = IM = \sum_{k=1}^{n} \lambda_k |v_k\rangle\langle w_k| ,$$

which is called the spectral decomposition of M.

Aside: When working with the stochastic six-vertex model

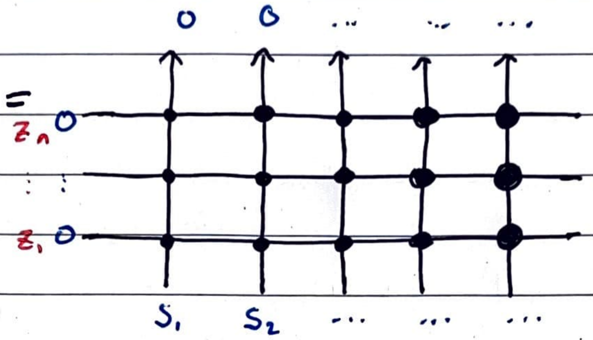
$$1,\ 1,\ B,\ qB,\ 1-B,\ 1-qB, \quad \text{where} \quad B = \frac{1-x/y}{1-qx/y} ,$$

the choice $0<q<1$, $0<x<y<1$ will yield a stochastic matrix.

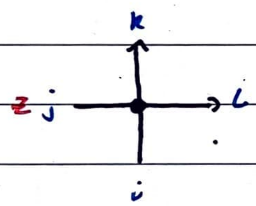
§6. Eigenvectors of $G_{S/T}$

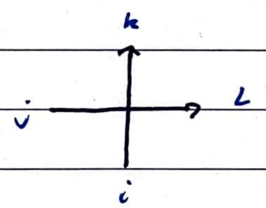
Definition Fix a finite string $S = (S_1, S_2, \dots)$.
Assume $|S| = n \geq 1$.
Fix also an alphabet $(z_1, \dots, z_n) \in \mathbb{R}^n$.

We define

$F_S(z_1, \dots, z_n) =$ 

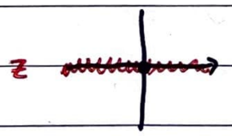
where we have a new normalization:



$=$ 

$\left(\dfrac{1 - qz/y}{1 - z/y} \right)$

In particular, this normalisation makes:



$= 1$

Theorem Fix a string $T = (T_1, T_2, \dots)$. Then, assuming certain convergence conditions,

$$\sum_S G_{S/T}(x_1, \dots, x_n)\, F_S(z_1, \dots, z_n) = \left(\prod_{i=1}^{N} \prod_{j=1}^{n} \frac{1 - q z_j / x_i}{1 - z_j / x_i} \right) F_T(z_1, \dots, z_n)$$