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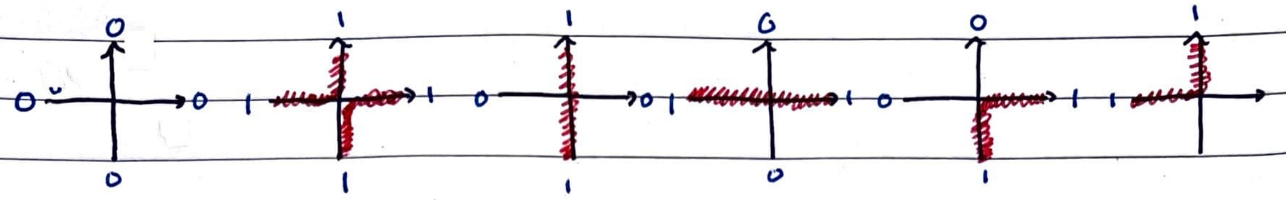
Exactly Solvable Models
Lecture 18



$1 \qquad 1 \qquad qB \qquad B \qquad 1-qB \qquad 1-B$

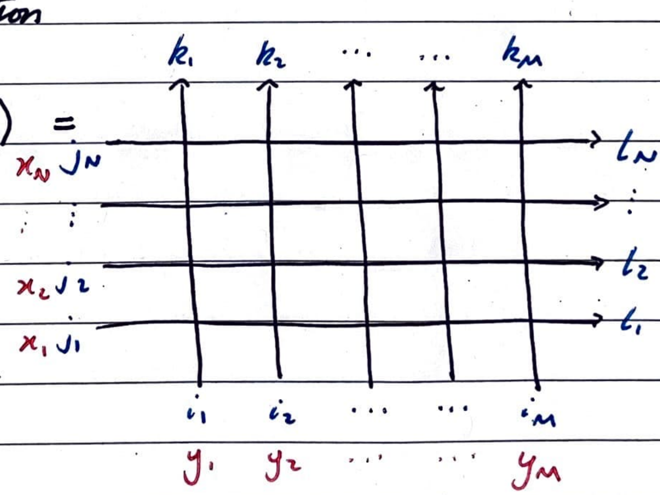
where $B = \dfrac{x/y - 1}{qx/y - 1}$

§3. Distribution functions

Fix two integers $M, N \geq 1$.
Fix also two vectors $\vec{i} = (i_1, \dots, i_M) \in \{0,1\}^M$
$\vec{j} = (j_1, \dots, j_N) \in \{0,1\}^N$

We define the function

$Z(\vec{i}, \vec{j} ; \vec{k}, \vec{l}) =$



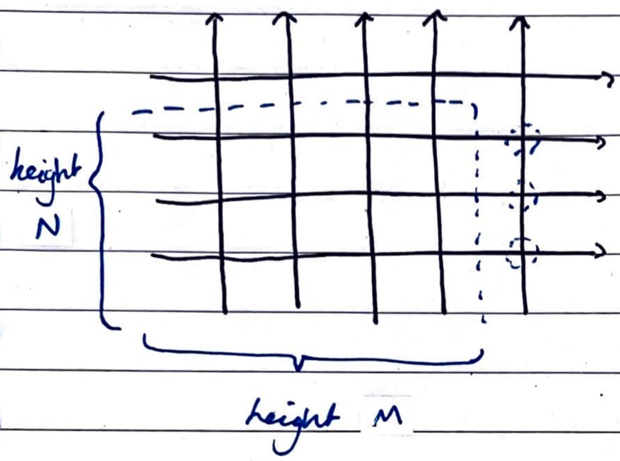
Consider the Markov process of paths in $\mathbb{N}^2$.
Then,

$$Z(\vec{i}, \vec{j} ; \vec{k}, \vec{l}) = \mathbb{P}((\vec{i}, \vec{j}) \to (\vec{k}, \vec{l})),$$

which is the probability of observing the path configuration

$(k_1, \dots, k_M, l_N, \dots, l_1)$

as one reads along a dotted path as indicated here:



Now we define a key random variable, the height function.

Definition Let $\mathcal{C}$ be a random configuration in the quadrant subject to some initial conditions.

Fix $M, N \geq 1$.
Define

$h(M,N)$ = # of paths in the configuration which interact with any of the points in the set $\{(M+1,1), (M+1,2), \dots, (M+1,N)\}$

The distribution of $h(M,N)$ is given by

$$\mathbb{P}(h(M,N) = p) = \sum_{\substack{\vec{k} \in \{0,1\}^M \\ \vec{l} \in \{0,1\}^N}} \mathbb{1}_{|\vec{l}|=p} \, Z(\vec{i}, \vec{j}, \vec{k}, \vec{l})$$

where $|\vec{l}| = \sum_{a=1}^{N} l_a$.

The q-moment of the height function is given by

$$\mathbb{E}\left(q^{h(M,N)}\right) = \sum_{p=0}^{n} q^{p} \cdot \mathbb{P}(h(M,N) = p)$$

$$= \sum_{\substack{\vec{k} \in \{0,1\}^M \\ \vec{l} \in \{0,1\}^N}} q^{|\vec{l}|} Z(\vec{i}, \vec{j}, \vec{k}, \vec{l}).$$