Gypsy Akhyar
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Exactly Solvable Models
Lecture 17

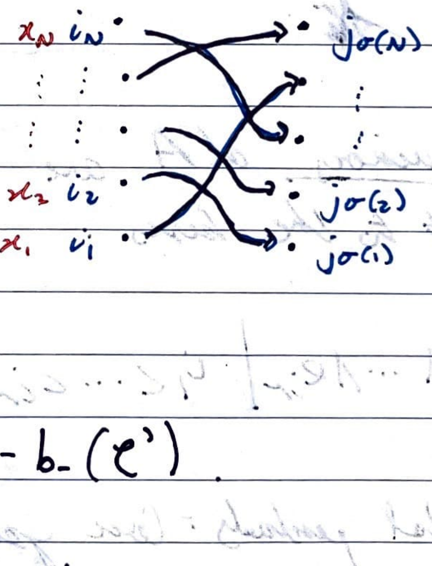
Last time:

$$\tilde{\tilde{R}}_{ab}(x/y) = \begin{pmatrix} 1 & & & \\ & b_+ & c_+ & \\ & c_- & b_- & \\ & & & 1 \end{pmatrix}_{ab}$$

where $b = b_+ = \dfrac{(x/y - 1)q^{\frac{1}{2}}}{qx/y - 1}$, $c_+ = \dfrac{q-1}{qx/y - 1}$, $c_- = \dfrac{(q-1)(x/y)}{qx/y - 1}$

**Lemma** For $\mathcal{C}$ and $\mathcal{C}'$ being configurations of

$$Z_\sigma(i_1, \dots, i_N; j_1, \dots, j_N) =$$




we have that

$$b_+(\mathcal{C}) - b_-(\mathcal{C}) = b_+(\mathcal{C}') - b_-(\mathcal{C}').$$

**Proof.** Use the path formulation of the model. Let $(i_1, \dots, i_N)$ and $(j_1, \dots, j_N)$ be chosen such that

$$\sum_{k=1}^{N} i_k = \sum_{k=1}^{N} j_k = N - 2p,$$ where $p$ is the number of paths, or you may which to understand it as the number of $-1$s.

Then, a total of $p$ paths enter the lattice via the left edges and exit via the right.

Let the entering vertical coordinates be

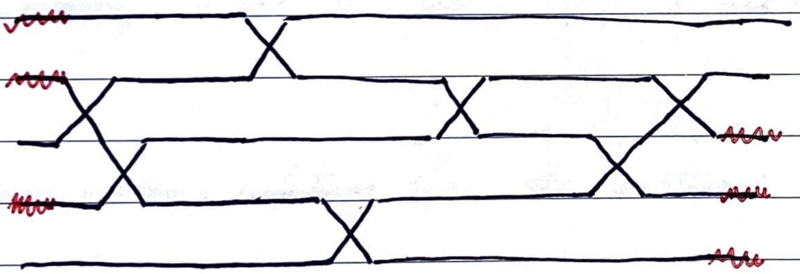
$$1 \le \alpha_1 < \cdots < \alpha_p \le N$$

and let the outgoing vertical coordinates be

$$1 \le \beta_1 < \cdots < \beta_p \le N.$$

Define $\Delta = \sum_{k=1}^{p} (\beta_k - \alpha_k)$, which is the total difference in the heights.

For example, for the picture



we have

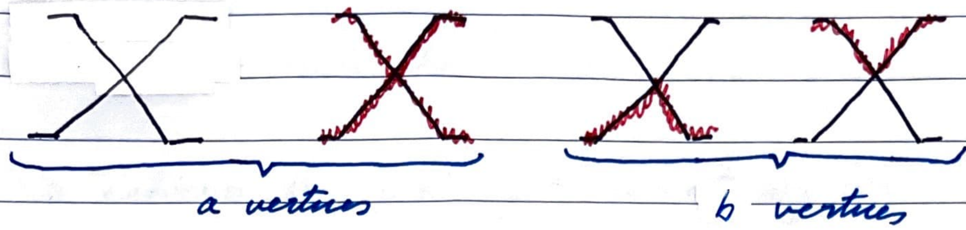
$\alpha_1 = 2$ $\quad \beta_1 = 1$ $\quad p = 3$

$\alpha_2 = 4$ $\quad \beta_2 = 2$ $\quad \Delta = (3-5) + (2-4) + (1-2)$

$\alpha_3 = 5$ $\quad \beta_3 = 3$ $\quad = -2 - 2 - 1$

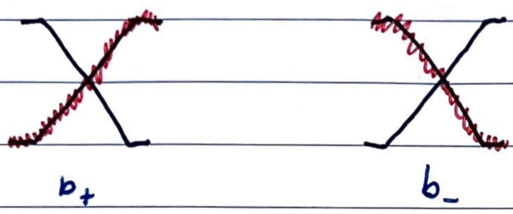
$= -5$

Now, consider one such path. Notice that the a and c vertices do not change this height:

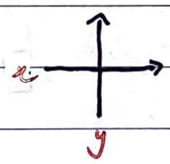


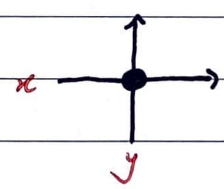

In the a-vertices, each particle may be thought as bouncing, and so the height of each particle stays constant.

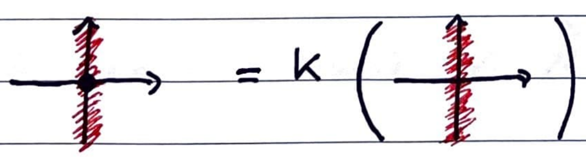
On the other hand, the b-vertices change the height:



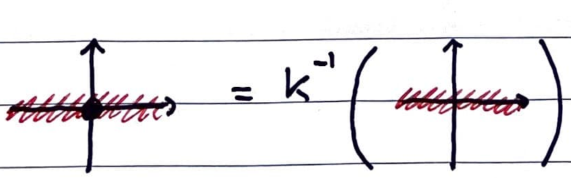

So one concludes that $\Delta = b_+(\mathcal{C}) - b_-(\mathcal{C})$ must be an invariant between configurations. $\square$

Corollary If the vertices  satisfy the Yang-Baxter equation

and unitarity relations then the vertices  where




and




and other vertices the same also satisfy the same equations.

We shall now make the choice $\underline{\kappa = q^{\frac{1}{2}}}$.

This achieves the goal of desymmetrising the $b$ vertices.

In particular, we have shown that the R-matrix

$$\tilde{\tilde{R}}(x/y) = \begin{pmatrix} 1 & & & \\ & b_+ & c_+ & \\ & c_- & b_- & \\ & & & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & \\ & \frac{q(x/y - 1)}{qx/y - 1} & \frac{q-1}{qx/y - 1} & \\ & \frac{(q-1)x/y}{qx/y - 1} & \frac{x/y - 1}{qx/y - 1} & \\ & & & 1 \end{pmatrix}$$

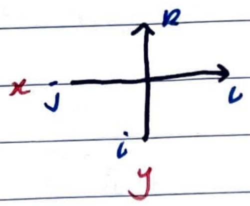
satisfies the Yang-Baxter equation and unitarity relations.

__Fact__ The row sums of $\tilde{\tilde{R}}(x/y)$ are all equal to 1.

Furthermore, provided that we make choices for $x, y, q$ such that all entries of the matrix are in the real interval $[0,1]$ then $\tilde{\tilde{R}}(x/y)$ is a stochastic matrix.

## §2. Stochastic Six-Vertex model in a quadrant

Let's summarise the facts that we have.

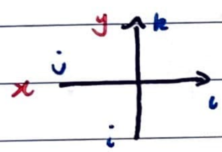
Let 

denote a generic vertex with $i, j, k, l \in \{0,1\}$

Here,

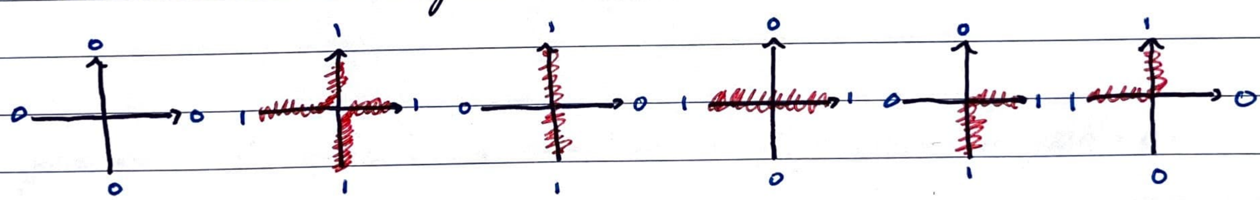
0 denotes the absence of a path.
1 denotes the presence of a path.

We have the ice rule, or conservation condition:



$= 0$ unless $i + j = k + l$.

When conservation is satisfied, our six vertices are:

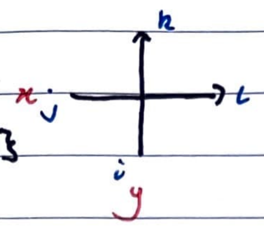



| | | | | | |
|---|---|---|---|---|---|
| $1$ | $1$ | $\dfrac{q(x/y-1)}{qx/y-1}$ | $\dfrac{x/y-1}{qx/y-1}$ | $\dfrac{q-1}{qx/y-1}$ | $\dfrac{(q-1)(x/y)}{qx/y-1}$ |
| $a$ | $a$ | $b_+$ | $b_-$ | $c_+$ | $c_-$ |

These weights satisfy the Yang-Baxter equation, the unitarity relations and stochasticity.

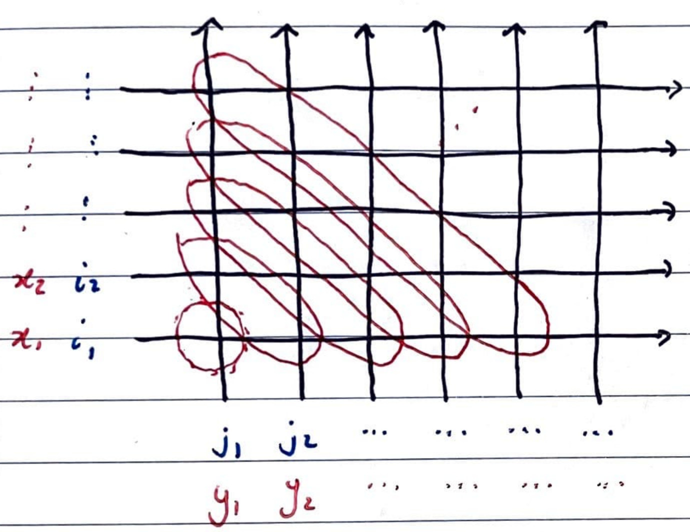
Stochasticity is the fact that for any choice $i, j \in \{0,1\}$ then

$$\sum_{k,l\in\{0,1\}}$$




$= 1$.

We construct a Markov process on the quadrant $\mathbb{N}^2$ by sampling in a diagonal fashion:




This process will give us a random configuration of paths on $\mathbb{N}^2$.