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Exactly Solvable Models
Lecture 16

3. Integrable Probability

Goals:

(1) Construct Markov processes from exactly solvable models

(2) Compute observables / distributions using the power of exactly solvable models: this is in systems of finite size.

(3) Compute asymptotics / scaling limits of the objects in (2). Typically long-time asymptotics are of interest. There are links here with analysis and random matrix theory.

We will focus on:

(1) Stochastic six-vertex model

(2) Reduction of (1) to the asymmetric simple exclusion process. (ASEP)

Remark: six-vertex model is an equilibrium statistical mechanics model. In the models of integrable probability, there is a notion of time or evolution.

§3.1 Stochastic R-matrix

We start from the R-matrix of the six-vertex model and perform modifications of it.

We still want the modified matrix to satisfy the Yang-Baxter equation but our aim is to get a stochastic matrix (all rows sum to 1).

$$R_{ab}(u,v) = \begin{pmatrix} 1 & & & \\ & \dfrac{e^{u-v} - e^{v-u}}{e^{u-v+\gamma} - e^{v-u-\gamma}} & \dfrac{e^{\gamma} - e^{-\gamma}}{e^{u-v+\gamma} - e^{v-u-\gamma}} & \\ & & & \\ & & & 1 \end{pmatrix}$$

Replace $e^{2u} = x$, $e^{2v} = y$ and $e^{2\gamma} = q$.
We write:

$$\tilde{R}_{ab}(x,y) = \begin{pmatrix} 1 & & & \\ & b & c & \\ & c & b & \\ & & & 1 \end{pmatrix}_{ab}$$

where

$$b = b(x,y) = \frac{(x/y - 1)\,q^{\frac{1}{2}}}{(qx/y - 1)}$$

$$c = c(x,y) = \frac{(q-1)(x/y)^{\frac{1}{2}}}{(qx/y - 1)}$$

We will 'desymmetrise' the b and c weights such that the 2^{nd} and 3^{rd} rows sum to 1.

a) $b \nearrow b_+$, $b \searrow b_-$ b) $c \nearrow c_+$, $c \searrow c_-$

We are aiming for:

$$\begin{pmatrix} 1 & & & \\ & b_+ & c_+ & \\ & c_- & b_- & \\ & & & 1 \end{pmatrix}$$

We start with (b). Define

$$d_a(x) = \begin{pmatrix} x^{-1/4} & \\ & x^{1/4} \end{pmatrix}_a \in End(V_a)$$

We take the Yang-Baxter equation and 'conjugate' by these matrices:

$$d_a(x)d_b(y)d_c(z)\tilde{R}_{ab}(x/y)\tilde{R}_{ac}(x/z)\tilde{R}_{bc}(y/z)d_a(x)^{-1}d_b(y)^{-1}d_c(z)^{-1}$$
$$=$$
$$LHS$$

and

$$d_a(x)d_b(y)d_c(z)\tilde{R}_{bc}(y/z)\tilde{R}_{ac}(x/z)\tilde{R}_{ab}(x/y)d_a(x)^{-1}d_b(y)^{-1}d_c(z)^{-1}$$
$$=$$
$$RHS$$

Write this as an equation for $\tilde{\tilde{R}}_{ab}$ where

$$\tilde{\tilde{R}}_{ab}(x/y) = d_a(x)d_b(y)\tilde{R}_{ab}(x/y)d_a(x)^{-1}d_b(y)^{-1}.$$

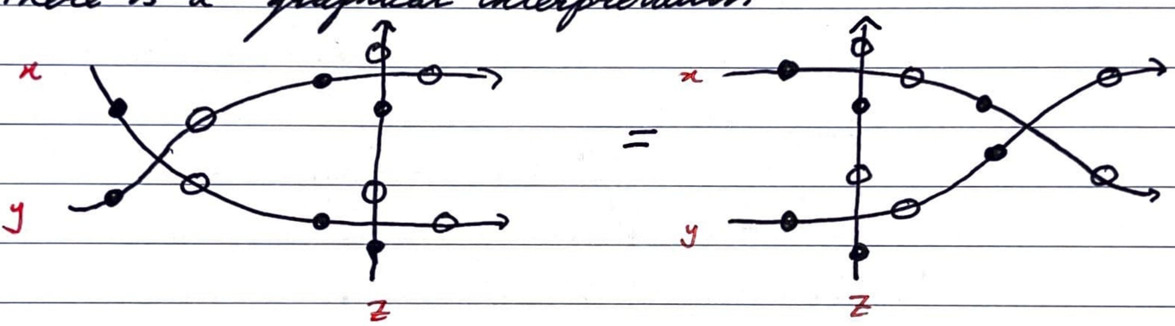
Then one has

$$\tilde{\tilde{R}}_{ab}(x/y)\tilde{\tilde{R}}_{ac}(x/z)\tilde{\tilde{R}}_{bc}(y/z) = \tilde{\tilde{R}}_{bc}(y/z)\tilde{\tilde{R}}_{ac}(x/z)\tilde{\tilde{R}}_{ab}(x/y)$$

Computing $\tilde{\tilde{R}}_{ab}(x/y)$ explicitly gives:

$$\tilde{\tilde{R}}_{ab}(x/y) = \begin{pmatrix} 1 & & & \\ & \frac{(x/y-1)q^{\frac{1}{2}}}{qx/y-1} & \frac{q-1}{qx/y-1} & \\ & \frac{(q-1)x/y}{qx/y-1} & \frac{(x/y-1)q^{\frac{1}{2}}}{qx/y-1} & \\ & & & 1 \end{pmatrix}$$

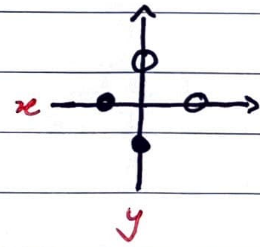
There is a graphical interpretation



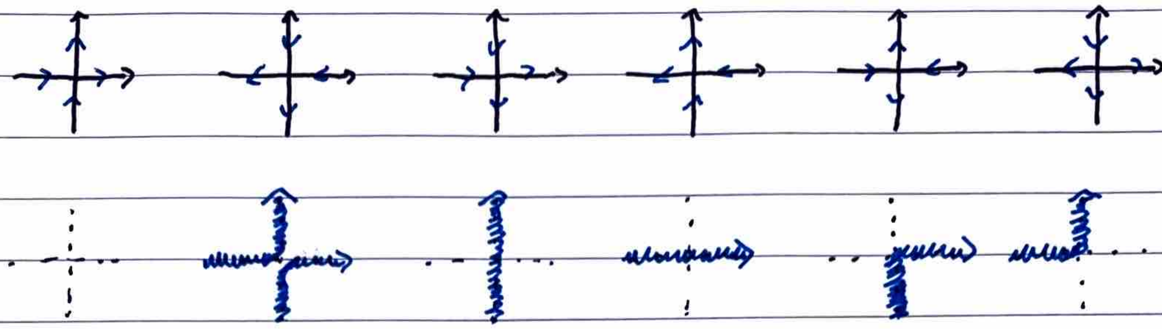
where

$$= \begin{pmatrix} x^{-\frac{1}{4}} & \\ & x^{1/4} \end{pmatrix}_a$$

$$= \begin{pmatrix} x^{1/4} & \\ & x^{-1/4} \end{pmatrix}_a$$

and $\tilde{\tilde{R}}(x/y) =$ 

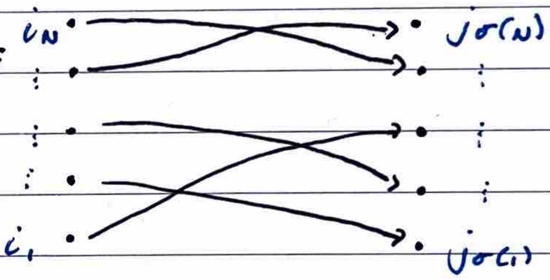
For step (a), we begin by drawing the six-vertex model vertices as follows:



(we go against the flow! cursed.)

Recall the definition

$Z_\sigma(i_1,\dots,i_N; j_1,\dots,j_N) :=$



Let C and C' denote two configurations of this partition function.

Write $b_k(C)$ as the # of b_k vertices in C where $k \in \{+,-\}$.

Lemma For any choice of configurations C, C', we have

$$b_+(C) - b_-(C) = b_+(C') - b_-(C').$$

In other words, the quantity $b_+(C) - b_-(C)$ is an invariant of configurations.