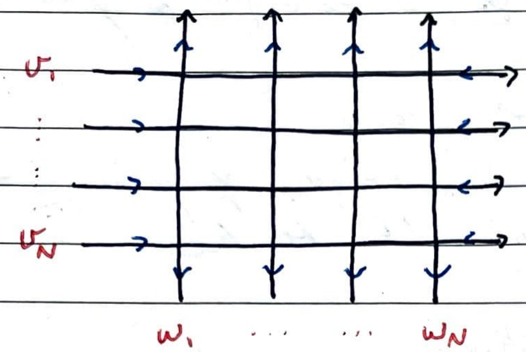
Gypsy Akhyar
27th August 2025

Exactly Solvable Models
Lecture 13 – 15 (P2)

Recall

$\underline{Z_N}$ =



is the domain wall partition function.

Also recall Korepin's conditions:

$\underline{(1)}$ Z_N is symmetric in $\{v_1, \dots, v_N\}$

$\underline{(2)}$ Z_N is a meromorphic function in w_N, with simple poles at the points $w_N = v_k + \gamma$

$\underline{(3)}$ $\mathrm{Res}[Z_N, w_N = v_N + \gamma]$

$$= -\sinh(\gamma) \prod_{k=1}^{N-1} \frac{\sinh(v_k - v_N - \gamma)}{\sinh(v_k - v_N)} \cdot \frac{\sinh(v_N - w_k)}{\sinh(v_N - w_k + \gamma)} \cdot Z_{N-1}$$

$\underline{(4)}$ $\lim\limits_{w_N \to \infty} Z_N = 0$

$\underline{(5)}$ $Z_1 = \dfrac{\sinh(\gamma)}{\sinh(v_1 - w_1 + \gamma)}$

Proof of Korepin's conditions.

We have already proved (1); it corresponds simply to the commutativity of B operators.

For (2), one notes that the w_N dependence comes from the right-most column of Z_N. It is clear that all weights in this column may only pick up simple poles at the specified points. To see this, we write the vertices that could appear:

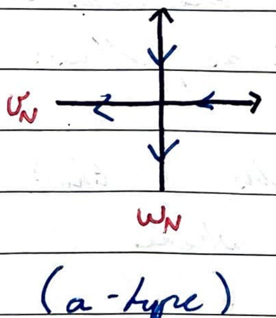
$$a = 1, \qquad b = \frac{\sinh(v_i - w_N)}{\sinh(v_i - w_N + \gamma)}, \qquad c = \frac{\sinh(\gamma)}{\sinh(v_i - w_N + \gamma)}$$

for $i \in \{1, \dots, N\}$ (corresponding to which row the vertex is placed).
A pole can only occur when

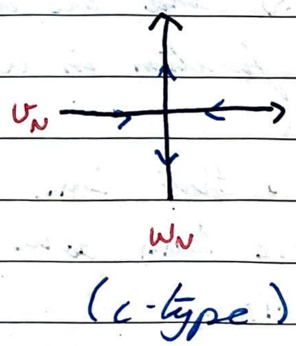
$$\sinh(v_i - w_N + \gamma) = 0,$$

which precisely happens when $w_N = v_i + \gamma \pmod{2\pi i}$.

For (3), we analyze the bottom-right vertex of Z_N. It takes the form



$= 1$ or

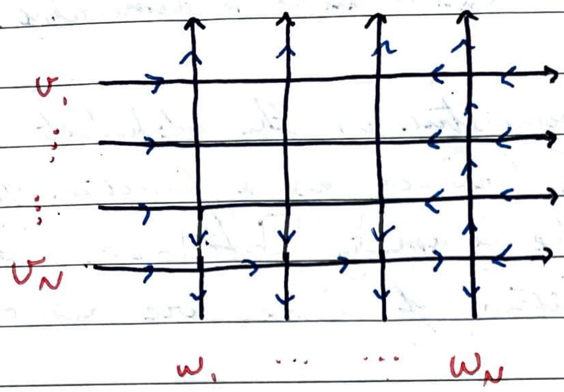


$$= \frac{\sinh(\gamma)}{\sinh(v_N - w_N + \gamma)}$$

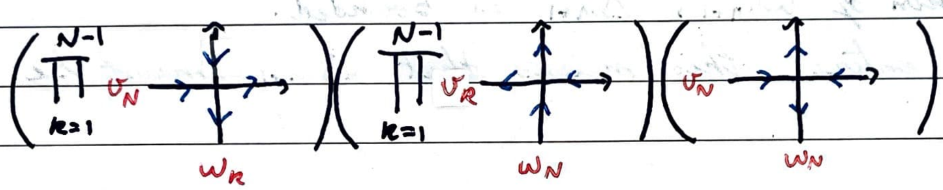
(a-type) (c-type).

Only the configurations that produce the c-type vertex will give a non-trivial contribution to the residue.

Further, imposing this choice in the partition function leads to the following freezing:



Evaluating the frozen region gives



$$= \left(\prod_{k=1}^{N-1} \frac{\sinh(v_N - w_k)}{\sinh(v_N - w_k + \gamma)}\right)\left(\prod_{k=1}^{N-1} \frac{\sinh(v_k - w_N)}{\sinh(v_k - w_N + \gamma)}\right)\left(\frac{\sinh(\gamma)}{\sinh(v_N - w_N + \gamma)}\right)$$

The residue of Z_N at $w_N = v_N + \gamma$ thus becomes

$$\lim_{w_N \to v_N + \gamma} Z_{N-1}(w_N - (v_N + \gamma)) \left(\prod_{k=1}^{N-1} \frac{\sinh(v_N - w_k)}{\sinh(v_N - w_k + \gamma)}\right)\left(\prod_{k=1}^{N-1} \frac{\sinh(v_k - w_N)}{\sinh(v_k - w_N + \gamma)}\right)\left(\frac{\sinh(\gamma)}{\sinh(v_N - w_N + \gamma)}\right)$$

$Z_{N-1} \times$

$$= \left(\prod_{k=1}^{N-1} \frac{\sinh(v_N - w_k)}{\sinh(v_N - w_k + \gamma)}\right)\left(\prod_{k=1}^{N-1} \frac{\sinh(v_k - (v_N + \gamma))}{\sinh(v_k - (v_N + \gamma) + \gamma)}\right) \sinh(\gamma) \lim_{w_N \to v_N + \gamma} \left(\frac{w_N - (v_N + \gamma)}{(-1)\sinh(w_N - (v_N + \gamma))}\right)$$

$$= -\sinh(\gamma) \prod_{k=1}^{N-1} \frac{\sinh(v_k - v_N - \gamma)}{\sinh(v_k - v_N)} \cdot \frac{\sinh(v_N - w_k)}{\sinh(v_N - w_k + \gamma)} \cdot Z_{N-1}.$$

(4) follows from noting that in every configuration of the rightmost column of Z_N, exactly one c-vertex appears.

The weight of the c-vertex has the claimed asymptotic behaviour (note that the numerator $\sinh(\gamma)$ is constant) and all other vertex weights are bounded as $w_N \to \infty$.

(5) is just the weight of a single c-vertex. $\square$

Theorem 2.5 Korepin's conditions determine the family $\{Z_N\}_{N \geq 1}$ uniquely.

Proof.

Suppose that there exists another family of functions $\{\tilde{Z}_N\}_{N \geq 1}$ that satisfies Korepin's conditions for each $N \geq 1$.

Define $\Delta_N = Z_N - \tilde{Z}_N$.

By (5), we have $\Delta_1 = 0$.
Assume that $\Delta_M = 0$ for some $M \geq 1$.

By (1), (2) and (3) and the inductive assumption, one sees that Δ_{M+1} is an entire function in w_{M+1}. We see this as follows:

(a) (2) tells us that Δ_{M+1} is meromorphic and tells us that the poles of Δ_{M+1} occur when $w_{M+1} = v_k + \gamma$ where $k \in \{1, \dots, M+1\}$.

(b) From here, (3) tells us that the residue of Z_{M+1} at $v_{M+1} + \gamma$ can be expressed in terms of Z_M and the residue of $\tilde{Z}_{M+1}$ at $v_{M+1} + \gamma$ can be expressed in terms of $\tilde{Z}_M$.

Yet, $Z_M = \tilde{Z}_M$.
So, we conclude that $\mathrm{Res}[\Delta_{M+1}, w_{M+1} = v_{M+1} + \gamma] = 0$.
So, the singularity of Δ_{M+1} at $w_{M+1} = v_{M+1} + \gamma$ is removable.

(c) (1) tells us that Δ_{M+1} is symmetric in the alphabet $\{v_1, \dots, v_{M+1}\}$.
Since all singularities of Δ_{M+1} (viewed as a function of w_{M+1}) are of the same form, the symmetry with (b) now tells us that all of the singularities of Δ_{M+1} are removable.

From these facts, one concludes that Δ_{M+1} is entire.

By (4), as a function of w_{M+1}, Δ_{M+1} is bounded.
Yet, since Δ_{M+1} is entire, this means that Δ_{M+1} must be constant.

Finally, also by (4), this constant must be 0.

Hence, all $\{\Delta_N\}_{N\geq 1}$ vanish by induction on N. $\square$

Theorem 2.6 (Izergin's determinant)

[Izergin 1986] found a remarkable determinant formula that satisfies all of Korepin's conditions:

$$Z_N = \left(\frac{\prod_{i,j=1}^{N} \sinh(v_i - w_j)}{\prod_{1\leq i<j\leq N} \sinh(v_i - v_j)\sinh(w_j - w_i)} \right) \det_{1\leq i,j\leq N} \left(\frac{\sinh(\gamma)}{\sinh(v_i - w_j)\sinh(v_i - w_j + \gamma)} \right)$$

Proof.

By the uniqueness of solutions to Korepin conditions, if we demonstrate that the above formula satisfies (1)-(5), it must be a faithful expression for Z_N.

(1) The determinant and the factor $\prod_{i<j} \sinh(v_i - v_j)$ are both

anti-symmetric under $v_i \leftrightarrow v_j$ while the other factors are symmetric. Hence, this antisymmetry cancels out leaving Z_N symmetric under $v_i \leftrightarrow v_j$.

(2) It is clear by the formula that Z_N is meromorphic in w_N. The poles at $w_N = v_i$ are removable by inspection of the formula, and the poles at $w_N = w_i$, $i \in \{1, \dots, N-1\}$ are removable as this will render two columns of the matrix equal, making the residue vanish.

(3) It is only the bottom-right entry of the determinant that contributes to the residue at $w_N = v_N + \gamma$. By Laplace expansion, we have that

$$\mathrm{Res}[Z_N ; w_N = v_N + \gamma]$$

$$= \left(\frac{\prod_{i,j=1}^{N} \sinh(v_i - w_j)}{\prod_{1 \le i<j \le N} \sinh(v_i - v_j) \sinh(w_j - w_i)} \right) \Bigg|_{w_N = v_N + \gamma} \mathrm{Res}\left[\frac{\sinh(\gamma)}{\sinh(v_N - w_N)\sinh(v_N - w_N + \gamma)} ; w_N = v_N + \gamma \right]$$

$$\cdot \det_{1 \le i,j \le N-1} \left(\frac{\sinh(\gamma)}{\sinh(v_i - w_j)\sinh(v_i - w_j + \gamma)} \right)$$

Yet, we have

$$\frac{\prod_{i,j=1}^{N} \sinh(v_i - w_j)}{\prod_{1\le i<j\le N} \sinh(v_i - v_j)\sinh(w_j - w_i)}$$

$$= \frac{\left(\prod_{i,j=1}^{N-1} \sinh(v_i - w_j)\right)\left(\prod_{i=1}^{N-1} \sinh(v_N - w_i)\sinh(v_i - w_N)\right)\left(\sinh(v_N - w_N)\right)}{\left(\prod_{1\le i<j\le N-1} \sinh(v_i - v_j)\sinh(w_j - w_i)\right)\left(\prod_{i=1}^{N-1} \sinh(v_i - v_N)\sinh(w_N - w_i)\right)}$$

At $w_N = v_N + \gamma$ this becomes

$$\frac{\left(\prod_{i=1}^{N-1} \sinh(v_N - w_i)\sinh(v_i - v_N - \gamma)\right)\left(\sinh(-\gamma)\right)}{\left(\prod_{i=1}^{N-1} \sinh(v_i - v_N)\sinh(v_N - w_i + \gamma)\right)}$$

$$\times \frac{\left(\prod_{i,j=1}^{N-1} \sinh(v_i - w_j)\right)}{\left(\prod_{1\le i<j\le N-1} \sinh(v_i - v_j)\sinh(w_j - w_i)\right)}$$

Next,

$$\mathrm{Res}\left[\frac{\sinh(\gamma)}{\sinh(v_N - w_N)\sinh(v_N - w_N + \gamma)} ;\ w_N = v_N + \gamma\right]$$

$$= \frac{\sinh(\gamma)}{\sinh(-\gamma)\,(-1)} = 1$$

So

$$\mathrm{Res}\,[Z_N\,;\,w_N = v_N + \gamma]$$

$$= -\sinh(\gamma) \prod_{i=1}^{N-1} \frac{\sinh(v_N - w_i)\sinh(v_i - v_N - \gamma)}{\sinh(v_i - v_N)\sinh(v_N - w_i + \gamma)}$$

$$\times \frac{\prod_{i,j=1}^{N-1} \sinh(v_i - w_j)}{\prod_{1\le i<j\le N-1} \sinh(v_i - v_j)\sinh(w_j - w_i)} \det_{1\le i,j\le N-1}\left(\frac{\sinh(\gamma)}{\sinh(v_i - w_j)\sinh(v_i - w_j + \gamma)}\right)$$

$$= -\sinh(\gamma) \prod_{i=1}^{N-1} \frac{\sinh(v_N - w_i)\sinh(v_i - v_N - \gamma)}{\sinh(v_i - v_N)\sinh(v_N - w_i + \gamma)} \; Z_{N-1}$$

as required.

(4) The limit $w_N \to \infty$ is taken by the degrees in the numerator and denominator with respect to e^{w_N}. For the numerator, one finds degree N in e^{w_N} and for the denominator degree $N+1$ in e^{w_N}. It immediately follows that $\lim_{w_N \to \infty} Z_N = 0$.

□

(5) Immediately verified.

VIP It is tempting to speculate that the Laplace expansion of Izergin's determinant should coincide with the sum-over-configurations definition of Z_N. This is false. The determinant expansion contains $N!$ terms, far fewer than the $\prod_{k=0}^{N-1} \frac{(3k+1)!}{(N+k)!}$ configurations of Z_N.

ASMs!