Gypsy Akhyar
15th August 2025

Exactly Solvable Models
Lecture 9

Spin subspaces

Define the 'total spin' operator

$$S : \mathbb{V} \longrightarrow \mathbb{V}$$

with action

$$S|s_1, \dots, s_n\rangle = \left(\frac{1}{2}\sum_{i=1}^{N} s_i\right)|s_1, \dots, s_n\rangle$$

for any basis vector $|s_1, \dots, s_n\rangle = |s_1\rangle \otimes \cdots \otimes |s_N\rangle_N \in \mathbb{V}$ and $s_i \in \{+,-\}$.

This leads to the definition

$$\mathbb{V}^{(k)} := \mathrm{Span}\left\{|s_1, \dots, s_n\rangle \,\middle|\, S|s_1, \dots, s_n\rangle = \left(\frac{N}{2} - k\right)|s_1, \dots, s_n\rangle\right\}$$

for all $0 \leq k \leq N$.

Facts : (a) $\dim(\mathbb{V}^{(k)}) = \binom{N}{k}$

(b) $\mathbb{V} = \bigoplus_{k=0}^{N} \mathbb{V}^{(k)}$

Claim $A(u) : \mathbb{V}^{(k)} \rightarrow \mathbb{V}^{(k)}$ $\quad B(u) : \mathbb{V}^{(k)} \longrightarrow \mathbb{V}^{(k+1)}$

$C(u) : \mathbb{V}^{(k)} \longrightarrow \mathbb{V}^{(k-1)}$ $\quad D(u) : \mathbb{V}^{(k)} \longrightarrow \mathbb{V}^{(k)}$

So $C(u)|+\rangle\!\rangle = 0$ and $B(u)|-\rangle\!\rangle = 0$

↳ highest weight vector ↳ lowest weight vector