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Exactly Solvable Models
Lecture 8

What we want (RHS of $\mathbb{P}_{m+1}$):

$$a(u)\prod_{k=1}^{m+1} f(v_k,u)\,|\underbrace{\{v_1,\dots,v_{m+1}\}\rangle\!\rangle} - \sum_{i=1}^{m+1} a(v_i)\,g(v_i,u)\prod_{k\neq i}^{m+1} f(v_k,v_i)\,|\{u,v_1,\dots,\hat{v}_k,\dots v_{m+1}\}\rangle\!\rangle$$

A vector of this form is called a Bethe vector.

What we currently have

$$a(u)\prod_{k=1}^{m+1} f(v_k,u)\,|\{v_1,\dots,v_{m+1}\}\rangle\!\rangle$$

$$-\sum_{i=1}^{m} a(v_i)\left[f(v_{m+1},u)\,g(v_i,u) - g(v_{m+1},u)\,g(v_i,v_{m+1})\right]$$

$$\prod_{k\neq i}^{m} f(v_k,v_i)\,|\{u,v_1,\dots,\widehat{v_k},\dots,v_{m+1}\}\rangle\!\rangle$$

$$-\,a(v_{m+1})\,g(v_{m+1},u)\prod_{k\neq m+1} f(v_k,v_{m+1})\,|\{u,v_1,\dots,v_m\}\rangle$$

Notice that:

$$f(v_{m+1},u)\,g(v_i,u) - g(v_{m+1},u)\,g(v_i,v_{m+1})$$

$$= f(v_{m+1},v_i)\,g(v_i,u) \qquad \dots (*)$$

(One can prove this, say, by using the result that $\sinh(x)=\frac{1}{2}(e^x-e^{-x})$)

Note that we will see $(*)$ reappear at some point, as it is a single Yang-Baxter equation in the 6-vertex model.

With this identity, we get a perfect match of summands.
So, P_{m+1} is true and by induction on n we are done. $\square$

Exercise 1.12: Prove the relation (2) by similar means.

Using this Theorem 1.6, we have that

$$t(u)\,|\{v_1,\dots,v_n\}\rangle\!\rangle = (A(u)+D(u))\,|\{v_1,\dots,v_n\}\rangle\!\rangle$$

$$= \underbrace{\left[a(u)\prod_{k=1}^{n} f(v_k,u) + d(u)\prod_{k=1}^{n} f(u,v_k)\right]|\{v_1,\dots,v_n\}\rangle\!\rangle}_{\text{good term :)}}$$

$$\underbrace{- \sum_{i=1}^{n}\left[a(v_i)\,g(v_i,u)\prod_{k\neq i}^{n} f(v_k,v_i) + d(v_i)\,g(u,v_i)\prod_{k\neq i}^{n} f(v_i,v_k)\right] |\{u,v_1,\dots,\hat{v}_i,\dots,v_n\}\rangle\!\rangle}_{\text{bad terms!!!}}$$

We note that we can make the bad terms vanish for particular choices of $v_1,\dots,v_n$. In particular, note that

$$g(u,v_i) = -g(v_i,u).$$

Therefore, the bad terms will vanish provided that

$$\underbrace{a(v_i)\prod_{k\neq i}^{n} f(v_k,v_i) = d(v_i)\prod_{k\neq i}^{n} f(v_i,v_k)}_{\text{these are the Bethe equations.}}$$

for all $1 \le i \le n$.

More explicitly,

$$\sinh^N\left(v_i + \frac{\gamma}{2}\right) \prod_{k \neq i}^{n} \sinh(v_k - v_i + \gamma) + (-1)^n \sinh^N\left(v_i - \frac{\gamma}{2}\right) \prod_{k \neq i}^{n} \sinh(v_i - v_k + \gamma)$$

$$=$$

$$0$$

for all $1 \leq i \leq n$.

Corollary 1.7 Assume that $\{v_1, \ldots, v_n\}$ is a solution to the Bethe equations. Then,

$$t(u) \, |\{v_1, \ldots, v_n\}\rangle\rangle = \underbrace{\left[a(u) \prod_{k=1}^{n} f(v_k, u) + d(u) \prod_{k=1}^{n} f(u, v_k) \right]}_{\text{eigenvalues of } t(u)} |\{v_1, \ldots, v_n\}\rangle\rangle$$

§ 1.12 Further Prospects

(1) Is the ABA complete? (i.e does one generate all eigenvectors and eigenvalues in this way?)

(2) What happens to the eigenvalues of $t(u)$ as $n, N \to \infty$?
(The usual approach is to take N even and consider $n = \frac{N}{2}$)
This is called the thermodynamic limit.

(3) The left eigenvectors of $t(u)$ take the form

$$\langle\langle + | C(v_1) \cdots C(v_n) | := \langle\langle \{v_1, \ldots, v_n\} | .$$

What is the norm-squared of the Bethe eigenvector?

It turns out that we get

$$\langle\!\langle \{v_1, \dots, v_n\} | \{v_1, \dots, v_n\} \rangle\!\rangle = \det(M)$$

for some matrix M (this is the Gaudin determinant)

(4) Calculating the correlation functions

$$\langle\!\langle \{u_1, \dots u_n\} | \sigma^+_{k_1} \sigma^-_{k_2} | \{u_1, \dots, u_n\} \rangle\!\rangle$$

(I have written $u_1, \dots, u_n$ but I mean $v_1, \dots, v_n$)

and

(not that it really matters)

$$\langle\!\langle \{u_1, \dots, u_n\} | \sigma^z_{k_1} \sigma^z_{k_2} | \{u_1, \dots, u_n\} \rangle\!\rangle$$

and considering $n, N \to \infty$. We wish to understand their asymptotic dependence on $|k_1 - k_2|$.

"The hard work just begins!"

– Ludvig Faddeev