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Exactly Solvable Models
Lecture 7

Recap: the pseudo-vacuum was defined as follows

$$|+\rangle\rangle := |+\rangle_1 \otimes \cdots \otimes |+\rangle_N$$

We proved that

$$A(u)|+\rangle\rangle = a(u)|+\rangle\rangle, \qquad a(u) = \sinh^N\left(u + \tfrac{\gamma}{2}\right)$$

$$D(u)|+\rangle\rangle = d(u)|+\rangle\rangle, \qquad d(u) = \sinh^N\left(u - \tfrac{\gamma}{2}\right)$$

$$\Rightarrow t(u)|+\rangle\rangle = (a(u) + d(u))|+\rangle\rangle.$$

§1.11 Creation Operators

The ABA is the 'guess' or 'postulate' that all eigenvectors of $t(u)$ are given by the following form:

$$|\{v_1, \ldots, v_n\}\rangle\rangle := B(v_1) \cdots B(v_n)|+\rangle\rangle$$

where $\{v_1, \ldots, v_n\} \subseteq \mathbb{C}$ are pairwise distinct and $0 \leq n \leq N$.

VIP: The vector $|\{v_1, \ldots, v_n\}\rangle\rangle$ is symmetric under pairwise swaps of the v_is e.g. $(v_i, v_j) \longleftrightarrow (v_j, v_i)$.

(Intuition: The total action of the B operator is to lower the total spin by 1)

We shall require the following functions:

$$f(u,v) = \frac{\sinh(u-v+\gamma)}{\sinh(u-v)}, \qquad g(u,v) = \frac{\sinh(\gamma)}{\sinh(u-v)}$$

Theorem For arbitrary complex parameters $u, v_1, \dots, v_n \in \mathbb{C}$ (that are pairwise distinct), the following expressions are true:

(1) $A(u) | \{v_1, \dots, v_n\} \rangle\rangle$

$$= a(u) \prod_{k=1}^{n} f(v_k, u) | \{v_1, \dots, v_n\} \rangle\rangle - \sum_{i=1}^{n} a(v_i) g(v_i, u) \prod_{k \neq i}^{n} f(v_k, v_i) | \{u, v_1, \dots, \widehat{v_i}, \dots, v_n\} \rangle\rangle$$

(2) $D(u) | \{v_1, \dots, v_n\} \rangle\rangle$

$$= d(u) \prod_{k=1}^{n} f(u, v_k) | \{v_1, \dots, v_n\} \rangle\rangle - \sum_{i=1}^{n} d(v_i) g(u, v_i) \prod_{k \neq i}^{n} f(v_i, v_k) | \{u, v_1, \dots, \widehat{v_i}, \dots, v_n\} \rangle\rangle$$

VIP: These are very good formulas because their RHS only has n+1 terms and, in particular, not 2^n terms

Proof of (1).

By induction on n.
(See Faddeev's lecture notes for a heuristic explanation)

Begin by checking $n=1$. Recall that, previously, we derived the relation:

$$A(u)B(v) = f(v,u)B(v)A(u) - g(v,u)B(u)A(v).$$

So,

$$A(u)|\{v\}\rangle\rangle = A(u)B(v)|+\rangle\rangle$$

$$= f(v,u)B(v)A(u)|+\rangle\rangle - g(v,u)B(u)A(v)|+\rangle\rangle$$

$$= a(u)f(v,u)|\{v\}\rangle\rangle - a(v)g(v,u)|\{u\}\rangle\rangle .$$

This validates $n=1$.
Suppose the proposition has the name $\mathbb{P}_n$.
So, we validated $\mathbb{P}_1$.

Next, suppose that m is true for some $m \geq 1$.
Start from the LHS of $\mathbb{P}_{m+1}$:

$$A(u)|\{v_1,\dots,v_{m+1}\}\rangle\rangle$$

$$= A(u)B(v_{m+1})B(v_1)\cdots B(v_m)|+\rangle\rangle$$

$$= A(u)B(v_{m+1})|\{v_1,\dots,v_m\}\rangle\rangle$$

$$= f(v_{m+1},u)B(v_{m+1})A(u)|\{v_1,\dots,v_m\}\rangle\rangle - g(v_{m+1},u)B(u)A(v_{m+1})|\{v_1,\dots,v_m\}\rangle\rangle$$

$$= f(v_{m+1},u)B(v_{m+1})\Big[a(u)\prod_{k=1}^{m} f(v_k,u)|\{v_1,\dots,v_m\}\rangle\rangle - \sum_{i=1}^{m} a(v_i)g(v_i,u)$$
$$\prod_{k\neq i}^{m} f(v_k,v_i)|\{u,v_1,\dots,\hat{v}_i,\dots,v_m\}\rangle\rangle\Big]$$

$$- g(v_{m+1},u)B(u)\Big[a(v_{m+1})\prod_{k=1}^{m} f(v_k,v_{m+1})|\{v_1,\dots,v_m\}\rangle\rangle - \sum_{i=1}^{m} a(v_i)g(v_i,v_{m+1})$$
$$\prod_{k\neq i}^{m} f(v_k,v_i)|\{v_{m+1},v_1,\dots,\hat{v}_i,\dots,v_m\}\rangle\rangle\Big]$$

$$= a(u) \prod_{k=1}^{m+1} f(v_k,u) \,|\, \{v_1,\dots,v_{m+1}\}\rangle\rangle$$

$$- \sum_{i=1}^{m} f(v_{m+1},u)\, a(v_i)\, g(v_i,u) \prod_{k\neq i}^{m} f(v_k,v_i) \,|\, \{u,v_1,\dots,\hat{v}_i,\dots,v_{m+1}\}\rangle\rangle$$

$$- g(v_{m+1},u)\, a(v_{m+1}) \prod_{k=1}^{m} f(v_k,v_{m+1}) \,|\, \{u,v_1,\dots,v_m\}\rangle\rangle$$

$$+ \sum_{i=1}^{m} g(v_{m+1},u)\, a(v_i)\, g(v_i,v_{m+1}) \prod_{k\neq i}^{m} f(v_k,v_i) \,|\, \{u,v_1,\dots,\hat{v}_i,\dots,v_{m+1}\}\rangle\rangle$$

$$= a(u) \prod_{k=1}^{m+1} f(v_k,u) \,|\, \{v_1,\dots,v_{m+1}\}\rangle\rangle$$

$$- \sum_{i=1}^{m} a(v_i) \left[f(v_{m+1},u)\, g(v_i,u) - g(v_{m+1},u)\, g(v_i,v_{m+1}) \right] \prod_{k\neq i}^{m} f(v_k,v_i)$$
$$|\, \{u,v_1,\dots,\hat{v}_i,\dots,v_{m+1}\}\rangle\rangle$$

$$- a(v_{m+1})\, g(v_{m+1},u) \prod_{k\neq m+1} f(v_k,v_{m+1}) \,|\, \{u,v_1,\dots,v_m\}\rangle\rangle .$$

TO BE CONTINUED... (in the next lecture!)