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Exactly Solvable Models
Lecture 6

Assignment given at the end of week 5 and it will be difficult.

§1.9 Algebraic Bethe Ansatz (ABA)

We have now proved that $[t(u), t(v)] = 0$ for all $u, v \in \mathbb{C}$.
Recall that

$$t(u) = tr_a(T_a(u))$$

$$= tr_a \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a$$

$$= A(u) + D(u).$$

So, we have in fact proven that $[A(u)+D(u), A(u)+D(u)] = 0$.

The ABA is a constructive procedure for computing eigenvectors and eigenvalues of $t(u)$.

The first step is to work out algebraic relations of

$$\{A(*), B(*), C(*), D(*)\}.$$

This set of elements with these algebraic relations has a name: Yang-Baxter Algebra.

The transfer matrix lives inside of this algebra.

Recall the RTT relation:

$$R_{ab}(u,v)\,T_a(u)\,T_b(v) = T_b(v)\,T_a(u)\,R_{ab}(u,v).$$

We will write this much more explicitly.
The LHS is given by

$$R_{ab}(u,v)\begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}_a \begin{pmatrix} A(v) & B(v) \\ C(v) & D(v) \end{pmatrix}_b$$

$$= R_{ab}(u,v)\left(\begin{array}{cc|cc} A(u)A(v) & A(u)B(v) & B(u)A(v) & B(u)B(v) \\ A(u)C(v) & A(u)D(v) & B(u)C(v) & B(u)D(v) \\ \hline C(u)A(v) & C(u)B(v) & D(u)A(v) & D(u)B(v) \\ C(u)C(v) & C(u)D(v) & D(u)C(v) & D(u)D(v) \end{array}\right),$$

where

$$R_{ab}(u,v) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \dfrac{\sinh(u-v)}{\sinh(u-v+\gamma)} & \dfrac{\sinh(\gamma)}{\sinh(u-v+\gamma)} & 0 \\ 0 & \dfrac{\sinh(\gamma)}{\sinh(u-v+\gamma)} & \dfrac{\sinh(u-v)}{\sinh(u-v+\gamma)} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The RHS is similar, except that we use the fact that

$$T_b(v)\,T_a(u) = P_{ab}\,T_a(v)\,T_b(u)\,P_{ab}$$

We get

$$T_b(v)\,T_a(u) = \left(\begin{array}{cc:cc} A(v)A(u) & B(v)A(u) & A(v)B(u) & B(v)B(u) \\ C(v)A(u) & D(v)A(u) & C(v)B(u) & D(v)B(u) \\ \hline A(v)C(u) & B(v)C(u) & A(v)D(u) & B(v)D(u) \\ C(v)C(u) & D(v)C(u) & C(v)D(u) & D(v)D(u) \end{array}\right)$$

In this way, the RTT relation provides 16 bilinear relations for the entries of $T_a(u), T_b(v)$.

This is precisely the Yang-Baxter algebra.

Exercise: Starting from these relations, prove that

$$[A(u)+D(u),\, A(v)+D(v)] = 0 .$$

We will only need 3 of these relations:

(1) $[B(u), B(v)] = 0$

(2) $B(u)A(v) = \dfrac{\sinh(\gamma)}{\sinh(u-v+\gamma)} B(v)A(u) + \dfrac{\sinh(u-v)}{\sinh(u-v+\gamma)} A(v)B(u)$

(3) $\dfrac{\sinh(\gamma)}{\sinh(u-v+\gamma)} B(u)D(v) + \dfrac{\sinh(u-v)}{\sinh(u-v+\gamma)} D(u)B(v) = B(v)D(u)$

§1.10 The pseudo-vacuum

Theorem The pseudo-vacuum $|+\rangle\!\rangle = |+\rangle_1 \otimes \cdots \otimes |+\rangle_N = \left.\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}\right\} 2^N$

is a 'trivial' eigenvector of $t(u)$.

(This will play the role of a 'highest-weight' vector c.f highest weight spaces in Lie Algebras)

Proof.

We will compute explicitly $t(u)|+\rangle\!\rangle = A(u)|+\rangle\!\rangle + D(u)|+\rangle\!\rangle$.

$$A(u)|+\rangle\!\rangle = \langle +|_a L_{a1}(u) \cdots L_{aN}(u)|+\rangle_a |+\rangle_1 \cdots |+\rangle_N$$

(the $\langle +|_a \cdots |+\rangle_a$ bit is giving the $(1,1)$, top left, entry)

$$= \langle +|_a \big(L_{a1}(u)|+\rangle_1\big) \cdots \big(L_{aN}(u)|+\rangle_N\big)|+\rangle_a$$

$$= \langle +|_a \begin{pmatrix} \begin{pmatrix} \sinh(u+\frac{\gamma}{2}) & 0 \\ 0 & \sinh(u-\frac{\gamma}{2}) \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 & \begin{pmatrix} 0 & 0 \\ \sinh(\gamma) & 0 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \\ \begin{pmatrix} 0 & \sinh(\gamma) \\ 0 & 0 \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 & \begin{pmatrix} \sinh(u-\frac{\gamma}{2}) & 0 \\ 0 & \sinh(u+\frac{\gamma}{2}) \end{pmatrix}_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}_1 \end{pmatrix}_a$$

$$\vdots$$

$$\begin{pmatrix} \begin{pmatrix} \sinh(u+\frac{\gamma}{2}) & 0 \\ 0 & \sinh(u-\frac{\gamma}{2}) \end{pmatrix}_N \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N & \begin{pmatrix} 0 & 0 \\ \sinh(\gamma) & 0 \end{pmatrix}_N \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N \\ \begin{pmatrix} 0 & \sinh(\gamma) \\ 0 & 0 \end{pmatrix}_N \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N & \begin{pmatrix} \sinh(u-\frac{\gamma}{2}) & 0 \\ 0 & \sinh(u+\frac{\gamma}{2}) \end{pmatrix}_N \begin{pmatrix} 1 \\ 0 \end{pmatrix}_N \end{pmatrix}_a$$

$$|+\rangle_a$$

$$= \langle +|_a \begin{pmatrix} \sinh(u+\frac{\gamma}{2})|+\rangle_1 & \sinh(\gamma)|+\rangle_1 \\ 0 & \sinh(u-\frac{\gamma}{2})|+\rangle_1 \end{pmatrix}_a$$

$$\vdots$$

$$\begin{pmatrix} \sinh(u+\frac{\gamma}{2})|+\rangle_N & \sinh(\gamma)|+\rangle_N \\ 0 & \sinh(u-\frac{\gamma}{2})|+\rangle_N \end{pmatrix}_a$$

$$|+\rangle_a$$

So, we have $A(u)|+\rangle\!\rangle = \sinh^N(u+\frac{\gamma}{2})|+\rangle\!\rangle$

Exercise: Show that, similarly, $D(u)|+\rangle\!\rangle = \sinh^N(u-\frac{\gamma}{2})|+\rangle\!\rangle$.