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Exactly Solvable Models
Lecture 4

1.8 RLL equation and proof of commuting transfer matrices

Our operators $L_{ak}(u)$ and $T_a(u)$ live in the 'enlarged' space $V_a \otimes \mathbb{V}$. We now enlarge this further to include two auxiliary spaces $V_a \simeq V_b \simeq \mathbb{C}^2$. We study identities in $V_a \otimes V_b \otimes \mathbb{V}$.

Theorem 1.4 There exists a matrix $R_{ab}(u,v) \in \mathrm{End}(V_a \otimes V_b)$ that is non-singular and satisfies

$$R_{ab}(u,v)\, L_{ak}(u)\, L_{bk}(v) = L_{bk}(v)\, L_{ak}(u)\, R_{ab}(u,v)$$

(This is called the RLL equation).

Furthermore, we have an explicit expression for $R_{ab}(u,v)$:

$$R_{ab}(u,v) = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\sinh(u-v)}{\sinh(u-v+\gamma)} \end{pmatrix}_b & \begin{pmatrix} 0 & 0 \\ \dfrac{\sinh(\gamma)}{\sinh(u-v+\gamma)} & 0 \end{pmatrix}_b \\ \begin{pmatrix} 0 & \dfrac{\sinh(\gamma)}{\sinh(u-v+\gamma)} \\ 0 & 0 \end{pmatrix}_b & \begin{pmatrix} \dfrac{\sinh(u-v)}{\sinh(u-v+\gamma)} & 0 \\ 0 & 1 \end{pmatrix}_b \end{pmatrix}_a$$

(This is the R-matrix of the 6-vertex model of statistical mechanics).

An alternate version of this uses the following definitions

$$\check{R}_{ab}(u,v) = P_{ab}\, R_{ab}(u,v), \qquad \check{L}_{ak}(u) = P_{ak}\, L_{ak}(u).$$

Exercise: Show that, in terms of the above

$$\check{R}_{bc}(u,v)\check{L}_{ab}(u)\check{L}_{bc}(v) = \check{L}_{ab}(v)\check{L}_{bc}(u)\check{R}_{ab}(u,v).$$

(At this point, Michael did a mathematica tutorial)

Corollary The monodromy matrix satisfies

$$R_{ab}(u,v)T_a(u)T_b(v) = T_b(v)T_a(u)R_{ab}(u,v)$$

Proof. We start from the LHS and perform step-by-step manipulations.

$$\begin{aligned}
R_{ab}(u,v)T_a(u)T_b(v) &= R_{ab}(u,v)L_{a1}(u)\cdots L_{aN}(u)L_{b1}(v)\cdots L_{bN}(v) \\
&= R_{ab}(u,v)[L_{a1}(u)L_{b1}(v)][L_{a2}(u)\cdots L_{aN}(u)][L_{b2}(v)\cdots L_{bN}(v)] \\
&= L_{b1}(v)L_{a1}(u)R_{ab}(u,v)[L_{a2}(u)\cdots L_{aN}(u)][L_{b2}(v)\cdots L_{bN}(v)] \\
&= (L_{b1}(v)L_{a1}(u))\cdots(L_{bN}(v)L_{aN}(u))R_{ab}(u,v) \\
&\vdots \quad \text{(reverse ordering as we did in line 2)} \\
&= L_{b1}(v)\cdots L_{bN}(v)L_{a1}(u)\cdots L_{aN}(u)R_{ab}(u,v) \\
&= T_b(v)T_a(u)R_{ab}(u,v) \qquad \square
\end{aligned}$$

Proof of Theorem 1.1 We shall use the following property (called the unitarity equation) of $R_{ab}(u,v)$:

$$R_{ba}(v,u)R_{ab}(u,v) = \mathrm{id} \in \mathrm{End}(V_a \otimes V_b)$$

Exercise: Show that $\check{R}_{ab}(v,u)\check{R}_{ab}(u,v) = \mathrm{id}$.

We have

it doesn't work here because $T_a(u), T_b(v) \notin \mathrm{End}(V_a \otimes V_b)$.

$$t(u)t(v) = tr_a\, tr_b\left(T_a(u)T_b(v)\right)$$

$$= tr_a\, tr_b\left(\mathbb{1}\, T_a(u)\, T_b(v)\right)$$

$$= tr_a\, tr_b\left(R_{ba}(v,u)\, R_{ab}(u,v)\, T_a(u)\, T_b(v)\right)$$

$$= tr_a\, tr_b\left(\left[R_{ba}(v,u)\, T_b(v) T_a(u)\right]\left[R_{ab}(u,v)\right]\right)$$

$$= tr_a\, tr_b\left(\left[R_{ab}(u,v)\right]\left[R_{ba}(v,u)\, T_b(v)\, T_a(u)\right]\right)$$

it is non-trivial why cyclicity works here, but it does. It is something to do with $R_{ab} \in \mathrm{End}(V_a \otimes V_b)$.

$$= tr_a\, tr_b\left(\mathbb{1}\, T_b(v)\, T_a(u)\right)$$

$$= tr_a\, tr_b\left(T_b(v)\, T_a(u)\right)$$

$$= t(v)\, t(u) \qquad \square$$