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Exactly Solvable Models
Filling in details of lecture 3

Define $T_a^N(u) = L_{a1}(u) \cdots L_{aN}(u)$.
Recall that

$$L_{aN}(u) = \begin{pmatrix} \begin{pmatrix} \sinh(u+\frac{\gamma}{2}) & \\ & \sinh(u-\frac{\gamma}{2}) \end{pmatrix}_N & \begin{pmatrix} & \\ \sinh(\gamma) & \end{pmatrix}_N \\ \begin{pmatrix} & \sinh(\gamma) \\ & \end{pmatrix}_N & \begin{pmatrix} \sinh(u-\frac{\gamma}{2}) & \\ & \sinh(u+\frac{\gamma}{2}) \end{pmatrix}_N \end{pmatrix}_a$$

So,

$$\frac{dL_{aN}(u)}{du} = \begin{pmatrix} \begin{pmatrix} \cosh(u+\frac{\gamma}{2}) & \\ & \cosh(u-\frac{\gamma}{2}) \end{pmatrix}_N & \\ & \begin{pmatrix} \cosh(u-\frac{\gamma}{2}) & \\ & \cosh(u+\frac{\gamma}{2}) \end{pmatrix}_N \end{pmatrix}_a$$

Substituting $u = \frac{\gamma}{2}$ gives

$$L_{aN}(u)\Big|_{u=\frac{\gamma}{2}} = \sinh(\gamma)\, P_{aN}, \qquad \frac{dL_{aN}(u)}{du}\Big|_{u=\frac{\gamma}{2}} = \begin{pmatrix} \begin{pmatrix} \cosh(\gamma) & \\ & 1 \end{pmatrix}_N & \\ & \begin{pmatrix} 1 & \\ & \cosh(\gamma) \end{pmatrix}_N \end{pmatrix}$$

Define $M_{aN} = \begin{pmatrix} \begin{pmatrix} \cosh(\gamma) & \\ & 1 \end{pmatrix}_N & \\ & \begin{pmatrix} 1 & \\ & \cosh(\gamma) \end{pmatrix}_N \end{pmatrix}_a$.

Using product rule, we thus get, if $N \geq 2$,

$$\frac{dT_a^N(u)}{du} = \frac{d}{du}\left[L_{a1}(u) \cdots L_{aN-1}(u)\right] L_{aN}(u) + L_{a1}(u) \cdots L_{aN-1}(u) \cdot \frac{d}{du}\left[L_{aN}(u)\right]$$

$$= \frac{dT_a^{N-1}(u)}{du} \cdot L_{aN}(u) + T_a^{N-1}(u) \cdot \frac{dL_{aN}(u)}{du}.$$

Substituting $u=\frac{\gamma}{2}$ gives:

$$\left.\frac{dT_a^N(u)}{du}\right|_{u=\frac{\gamma}{2}} = \left.\frac{dT_a^{N-1}(u)}{du}\right|_{u=\frac{\gamma}{2}} \cdot L_{aN}(u)\Big|_{u=\frac{\gamma}{2}} + T_a^{N-1}(u)\Big|_{u=\frac{\gamma}{2}} \cdot \left.\frac{dL_{aN}(u)}{du}\right|_{u=\frac{\gamma}{2}}$$

$$= \left.\frac{dT_a^{N-1}(u)}{du}\right|_{u=\frac{\gamma}{2}} \sinh(\gamma) P_{aN} + \sinh^{N-1}(\gamma) P_{a1} \cdots P_{aN} M_{aN}$$

Iterating this N-1 times, we get

$$\left.\frac{dT_a^N(u)}{du}\right|_{u=\frac{\gamma}{2}} = \left.\frac{dT_a(u)}{du}\right|_{u=\frac{\gamma}{2}} \sinh^{N-1}(\gamma) P_{a2} \cdots P_{aN} + \sum_{k=2}^{N} \sinh^{N-1}(\gamma) \Big[P_{a1} \cdots P_{a,k-1} \cdots M_{ak} P_{a,k+1} \cdots P_{aN} \Big]$$

$$= M_{a1} \cdot \sinh^{N-1}(\gamma) P_{a2} \cdots P_{aN} + \sum_{k=2}^{N} \sinh^{N-1}(\gamma) \left[P_{a1} \cdots P_{a,k-1} M_{ak} P_{a,k+1} \cdots P_{aN} \right]$$

$$= \sinh^{N-1}(\gamma) \sum_{k=1}^{N} \left[P_{a1} \cdots P_{a,k-1} M_{ak} P_{a,k+1} \cdots P_{aN} \right]$$

Next, let us find $\left.\frac{dt(u)}{du}\right|_{u=\frac{\gamma}{2}}$.

We have,

$$\left.\frac{dt(u)}{du}\right|_{u=\frac{\gamma}{2}} = \sinh^{N-1}(\gamma) \sum_{k=1}^{N} tr_a \left(P_{a1} \cdots P_{a,k-1} M_{ak} P_{a,k+1} \cdots P_{aN} \right)$$

When $k=1$, we have, in the summand,

$$
\begin{aligned}
M_{a1}P_{a2}\cdots P_{aN} &= M_{a1}P_{23}\cdots P_{2N}P_{a2} \\
&= P_{23}\cdots P_{2N}M_{a1}P_{a2} \\
&= P_{23}\cdots P_{2N}P_{a2}M_{21} \\
&= P_{23}\cdots P_{2N-1}P_{2N}P_{a2}M_{21} \\
&= P_{23}\cdots P_{2N-1}P_{aN}M_{N1}P_{2N} \\
&= P_{aN}P_{23}\cdots P_{2N-1}M_{N1}P_{2N} \\
&= P_{aN}M_{N1}P_{23}\cdots P_{2N-1}P_{2N}
\end{aligned}
$$

So,

$$tr_a(M_{a1}P_{a2}\cdots P_{aN}) = M_{N1}P_{23}\cdots P_{2N}$$

When $k\neq 1$,

$$
\begin{aligned}
&tr_a(P_{a1}\cdots P_{ak-1}M_{ak}P_{ak+1}\cdots P_{aN}) \\
&= tr_a(P_{12}\cdots P_{1,k-1}M_{1k}P_{1,k+1}\cdots P_{1N}P_{a1}) \\
&= P_{12}\cdots P_{1,k-1}M_{1k}P_{1,k+1}\cdots P_{1N}
\end{aligned}
$$

This gives the result that:

$$\left.\frac{dt(u)}{du}\right|_{u=\frac{\gamma}{2}} = \sinh^{N-1}\left[\sum_{k=2}^{N}\left(P_{12}\cdots P_{1,k-1}M_{1k}P_{1,k+1}\cdots P_{1N}\right) + M_{N1}P_{23}\cdots P_{2N}\right]$$

Next,

$$\sinh(\gamma) \frac{dt(u)}{du}\Big|_{u=\frac{\gamma}{2}} \quad t(\tfrac{\gamma}{2})^{-1}$$

$$= \left[\sum_{k=2}^{N} (P_{12} \cdots P_{1,k-1}) M_{1,k} (P_{1,k+1} \cdots P_{1,N}) + M_{N,1} P_{23} \cdots P_{2N}\right] (P_{1N} \cdots P_{12})$$

Now, we have, in the $k=1$ case:

$$\begin{aligned}
&M_{N,1} (P_{23} \cdots P_{2N})(P_{1N} \cdots P_{12}) \\
&= M_{N,1} (P_{23} \cdots P_{2N-1})(P_{2N} P_{1N})(P_{1N-1} \cdots P_{12}) \\
&= M_{N1} (P_{23} \cdots P_{2N-1})(P_{12} P_{2N})(P_{1N-1} \cdots P_{12}) \\
&= M_{N1} (P_{12} \cdots P_{1N-1})(P_{2N})(P_{1N-1} \cdots P_{12}) \\
&= M_{N1} (P_{12} \cdots P_{1N-1})(P_{1N-1} \cdots P_{13})(P_{2N} P_{12}) \\
&= M_{N1} P_{12} P_{2N} P_{12} \\
&= M_{N1} P_{1N}
\end{aligned}$$

In the $k \neq 1$ case, we have

$$\begin{aligned}
&(P_{12} \cdots P_{1k-1}) M_{1k} (P_{1k+1} \cdots P_{1N})(P_{1N} \cdots P_{12}) \\
&= (P_{12} \cdots P_{1k-1}) M_{1k} (P_{1k+1} \cdots P_{1N})(P_{1N} \cdots P_{1k+1})(P_{1k} \cdots P_{12}) \\
&= (P_{12} \cdots P_{1k-1}) M_{1k} (P_{1k} \cdots P_{12}) \\
&= (P_{12} \cdots P_{1k-2})(P_{1k-1} M_{1k})(P_{1k} \cdots P_{12}) \\
&= (P_{12} \cdots P_{1k-2})(M_{k-1k} P_{1k-1})(P_{1k} \cdots P_{12}) \\
&= M_{k-1,k} (P_{12} \cdots P_{1,k-1})(P_{1k} \cdots P_{12}) \\
&= M_{k-1,k} (P_{12} \cdots P_{1,k-1})(P_{1k})(P_{1k-1} \cdots P_{12})
\end{aligned}$$

Now, use the relation that $M_{ac} = P_{ab} M_{bc} P_{ab}$

So,

$$\begin{aligned}
&(P_{12} \cdots P_{1k-1}) M_{1k} (P_{1k+1} \cdots P_{1N})(P_{1N} \cdots P_{12}) \\
&= M_{k-1,k} (P_{12} \cdots P_{1,k-2})(P_{1,k-1} P_{1k} P_{1,k-1})(P_{1,k-2} \cdots P_{12}) \\
&= M_{k-1,k} (P_{12} \cdots P_{1,k-2}) P_{k,k-1} (P_{1,k-2} \cdots P_{12})
\end{aligned}$$

$$= M_{k-1,k}\, P_{k,k-1}.$$

So, we have found that

$$\sinh(\gamma) \left.\frac{d\,t(u)}{du}\right|_{u=\frac{\gamma}{2}} t\left(\tfrac{\gamma}{2}\right)^{-1}$$

$$= M_{N1} P_{1N} + \sum_{k=2}^{N} M_{k-1,k}\, P_{k,k-1}$$

$$= \sum_{k=1}^{N} M_{k-1,k}\, P_{k,k-1},$$ where we have made the identification $0 \equiv N$.

Now, recall that $P_{k,k-1} = P_{k-1,k}$, and so

$$M_{k-1,k}\, P_{k,k-1} = \begin{pmatrix} \begin{pmatrix} \cosh(\gamma) & \\ & 1 \end{pmatrix}_k & \begin{pmatrix} & \\ & \end{pmatrix}_k \\ \begin{pmatrix} & \\ & \end{pmatrix}_k & \begin{pmatrix} 1 & \\ & \cosh(\gamma) \end{pmatrix}_k \end{pmatrix}_{k-1} \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_k \end{pmatrix}_{k-1}$$

$$= \begin{pmatrix} \begin{pmatrix} \cosh(\gamma) & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & \cosh(\gamma) \end{pmatrix}_k \end{pmatrix}_{k-1}.$$

So,

$$\sinh(\gamma) \left.\frac{dt(u)}{du}\right|_{u=\frac{\gamma}{2}} t\left(\tfrac{\gamma}{2}\right)^{-1} = \sum_{k=1}^{N} \begin{pmatrix} \begin{pmatrix} \cosh(\gamma) & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & \cosh(\gamma) \end{pmatrix}_k \end{pmatrix}_{k-1}$$

as required.

Finally, we show that this expression is in fact equal to the Hamiltonian.

We have that: (defining $\Delta = e^{\gamma} + e^{-\gamma}$)

$$\left(\sigma^x_{k-1}\sigma^x_k + \sigma^y_{k-1}\sigma^y_k + \Delta\sigma^z_{k-1}\sigma^z_k\right)\frac{1}{2}$$

$$= \frac{1}{2}\left(\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_k \end{pmatrix}_{k-1} + \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_k \end{pmatrix}_{k-1} + \begin{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}_k \end{pmatrix}_{k-1} \right)$$

$$= \frac{1}{2} \begin{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & \Delta \end{pmatrix}_k \end{pmatrix}_{k-1}$$

$$= \begin{pmatrix} \begin{pmatrix} \cosh(\gamma) & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & \cosh(\gamma) \end{pmatrix}_k \end{pmatrix}_{k-1}$$

and so we are done.