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Exactly Solvable Models
Lecture 3

From last time:

$$L_{ak}\left(u = \frac{\gamma}{2}\right) = \sinh(\gamma) \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_k \end{pmatrix}_a$$

$$= \sinh(\gamma)\, P_{ak}$$

Properties of the permutation matrix

(1) $P_{ab} = P_{ba}$

(2) If $M_{ab} \in \mathrm{End}(V_a \otimes V_b)$ is an arbitrary matrix, then

$$P_{ab} M_{ab} = M_{ba} P_{ab}$$

In other words, pushing matrices past other matrices swaps the action on their component spaces.

(3) If $M_{ac} \in \mathrm{End}(V_a \otimes V_c)$, then

$$P_{ab} M_{ac} = M_{bc} P_{ab}$$

The proof is left as an exercise

From here, consider the $u = \frac{\gamma}{2}$ specialisation of the monodromy matrix:

$$T_a\left(u = \tfrac{\gamma}{2}\right) = L_{a_1}\left(\tfrac{\gamma}{2}\right) \cdots L_{aN}\left(\tfrac{\gamma}{2}\right)$$

$$= \sinh^N(\gamma)\, P_{a1} \cdots P_{aN} .$$

Taking the trace over V_a, we get the transfer matrix:

$$t\left(u = \tfrac{\gamma}{2}\right) = tr_a\left(T_a\left(\tfrac{\gamma}{2}\right)\right)$$

$$= \sinh^N(\gamma)\, tr_a\left(P_{a_1} \cdots P_{aN}\right)$$

By property (3) of permutation matrices, we may push P_{a1} past $P_{a2} \cdots P_{aN}$ so that

$$(P_{a_1})(P_{a2} \cdots P_{aN}) = (P_{12} \cdots P_{1N})\, P_{a1} .$$

Further, we have

$$tr_a(P_{a1}) = tr_a\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_1 & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_1 \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_1 & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_1 \end{pmatrix}_a$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_1 + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_1$$

$$= \mathbb{1}_1$$

So, $t\left(u = \frac{\gamma}{2}\right) = \sinh^N(\gamma)\, P_{12} \cdots P_{1N}$.

Note that $P_{12} \cdots P_{1N}$ is cyclically permuting by one unit all of the sites of the spin chain. In the literature, you thus might find this operator, commonly called U, be referred to as the 'cycle' or 'shift' operator.

Since the permutation matrices are self-inverse, notice

$$t\left(\tfrac{\gamma}{2}\right)^{-1} = \frac{P_{1N} \cdots P_{12}}{\sinh^N(\gamma)} .$$

Next, consider the derivative of the monodromy matrix $T_a(u)$ with respect to u, and then specialising at $u = \frac{\gamma}{2}$. (Note that derivatives of matrices are derivatives of their entries).

By the product rule, one gets:

$$\left.\frac{d\, T_a(u)}{du}\right|_{u=\frac{\gamma}{2}} = \sinh^{N-1}(\gamma) \sum_{k=1}^{N} P_{a1} \cdots P_{a,k-1} \cdot M_1 \cdot P_{a,k+1} \cdots P_{aN}$$

where
$$M_{ak} = \begin{pmatrix} \begin{pmatrix} \cosh(\gamma) & 0 \\ 0 & 1 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 1 & 0 \\ 0 & \cosh(\gamma) \end{pmatrix}_k \end{pmatrix}_a$$

(This derivation is an exercise)

Hence,

$$\left.\frac{d\, t(u)}{du}\right|_{u=\frac{\gamma}{2}} = \operatorname{tr}_a\left(\left.\frac{d\, T_a(u)}{du}\right|_{u=\frac{\gamma}{2}} \right)$$

$$= \sinh^{N-1}(\gamma)\left[\sum_{k=2}^{N}\left(P_{12}\cdots P_{1,k-1}M_{1k}P_{1,k+1}\cdots P_{1N}\right) + M_{N1}P_{23}\cdots P_{2N}\right]$$

(This derivation is an exercise; think about the $k=1$ and $k\neq 1$ cases seperately).

Combining (3) and (4), we get the following:

$$\sinh(\gamma)\left.\frac{dt(u)}{du}\right|_{u=\frac{\gamma}{2}} \cdot t\left(\frac{\gamma}{2}\right)^{-1}$$

$$= \sum_{k=1}^{N}\begin{pmatrix} \begin{pmatrix}\cosh(\gamma) & 0\\ 0 & 0\end{pmatrix}_k & \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}_k \\ \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}_k & \begin{pmatrix}0 & 0\\ 0 & \cosh(\gamma)\end{pmatrix}_k\end{pmatrix}_{k-1},$$

where we identify $0 \equiv N$.

Finally, note that

$$\sigma^x_{k-1}\sigma^x_k = \begin{pmatrix} \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}_k & \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_k \\ \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}_k & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}_k\end{pmatrix}_{k-1}$$

$$\sigma^y_{k-1}\sigma^y_k = \begin{pmatrix} \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}_k & \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}_k \\ \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}_k & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}_k\end{pmatrix}_{k-1}$$

$$\sigma^z_{k-1}\sigma^z_k = \begin{pmatrix} \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}_k & \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}_k \\ \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}_k & \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}_k\end{pmatrix}_{k-1}$$

from which, we get the result that

$$\sinh(\gamma) \left. \frac{dt(u)}{du} \right|_{u=\frac{\gamma}{2}} t\left(\tfrac{\gamma}{2}\right)^{-1}$$

$$= H$$

$$= \frac{1}{2} \sum_{k=1}^{N} \left(\sigma_{k-1}^{x} \sigma_{k}^{x} + \sigma_{k-1}^{y} \sigma_{k}^{y} + \Delta\, \sigma_{k-1}^{z} \sigma_{k}^{z} \right)$$

where $\Delta = e^{\gamma} + e^{-\gamma}$.

(This derivation is an exercise).

<u>Corollary 1.3</u> $[H, t(u)] = 0 \quad \forall u \in \mathbb{C}$

Proof. Plug in expression of H we just found into the commutator:

$$\sinh(\gamma) \frac{d}{dv} \left[[t(v) t(w)^{-1}, t(u)] \right] \Big|_{\substack{v=\frac{\gamma}{2} \\ u=\frac{\gamma}{2}}} = [H, t(u)] = 0.$$

Note that $[H, t(u)] = 0$ means $t(u)$ is a conserved quantity (commuting with the Hamiltonian is the definition of conserved quantity).