Gypsy Akhyar
29th July 2025

Exactly Solvable Models
Lecture 2

Transfer Matrices

We will build transfer matrices using a 'local' operator defined as:

$$L_k(u) = \left(\begin{array}{c:c} \sinh(u\mathbb{1} + \frac{\gamma}{2}\sigma_k^z) & \sigma_k^- \sinh(\gamma) \\ \hdashline \sigma_k^+ \sinh(\gamma) & \sinh(u\mathbb{1} - \frac{\gamma}{2}\sigma_k^z) \end{array}\right)$$

where $u, \gamma \in \mathbb{C}$ and $\sigma_k^\pm = \frac{1}{2}(\sigma_k^x \pm i\sigma_k^y)$.

$L_k(u)$ is called a Lax matrix.

VIP: $L_k(u)$ is a 2×2 matrix with entries that are $2^N \times 2^N$ matrices.

Exercise 1.5: Show that, alternatively, we can write

$$L_k(u) = \left(\begin{array}{c:c} \begin{pmatrix} \sinh(u+\frac{\gamma}{2}) & \\ & \sinh(u-\frac{\gamma}{2}) \end{pmatrix}_k & \begin{pmatrix} & \\ \sinh(\gamma) & \end{pmatrix}_k \\ \hdashline \begin{pmatrix} & \sinh(\gamma) \\ & \end{pmatrix}_k & \begin{pmatrix} \sinh(u-\frac{\gamma}{2}) & \\ & \sinh(u+\frac{\gamma}{2}) \end{pmatrix}_k \end{array}\right)$$

where the blank spaces correspond to 0s and the subscript k is shorthand for

$$A_k = \mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \underset{k^{th}\text{ position}}{A} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} \in \mathrm{End}(\mathcal{V}).$$

From here, define the matrix:

$$T(u) = L_1(u) \cdots L_N(u).$$

VIP: $T(u)$ is a 2×2 matrix with entries that are $2^N \times 2^N$ matrices.

We give names to the entries of the $T(u)$ matrix as follows:

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}.$$

Exercise 1.6: Compute explicitly $T(u)$ when $N=2$.

Finally, we arrive at the definition of the transfer matrix:

$$t(u) = A(u) + D(u) = \mathrm{tr}(T(u)).$$

VIP: This trace operator above is applied to the 'outer' 2×2 matrix but not to the 'inner' $2^N \times 2^N$ matrices.

<u>Theorem 1.1</u> The transfer matrices satisfy:

$$[t(u), t(v)] = 0, \qquad \forall u, v \in \mathbb{C}.$$

Here, note that γ is the same in both $t(u)$ and $t(v)$; it is universal.

Proof. This is a miracle. We will do it later.

Exercise 1.7: Check Theorem 1.1 for $N=2$ explicitly.

§ 1.7 Commuting quantities and H

Consider the dependence of $t(u)$ on e^u
After some thought, we see that:

$$t(u) = e^{-Nu} \sum_{k=0}^{N} e^{2ku} t^{(k)}$$

That is, we expand $t(u)$ over the basis $\{e^{2ku}\}$ and factor e^{-Nu} out of all the coefficients.

Note here that the coefficients $\{t^{(1)}, \dots, t^{(k)}, \dots, t^{(N)}\}$ are operators that in particular do not depend on e^u (but might still depend on e^{γ}, say).

Define $\mathrm{span}_{\mathbb{C}}\{t^{(0)}, \dots, t^{(N)}\}$ as the space of conserved quantities.

Corollary 1.2 For all $j, k \in \{0, \dots, N\}$, one has that

$$[t^{(j)}, t^{(k)}] = 0.$$

To progress towards H, we 'decorate' our L and T matrices with an extra label:

$$L_k(u) \equiv L_{ak}(u).$$

In this way, we think of $L_{ak}(u) \in \mathrm{End}(V_a \otimes V)$ where $V_a = V$, is called the auxiliary space.

The way we think about this is the following.
If A is a 2×2 matrix with entries that are $2^N \times 2^N$ matrices, we write

$$A \equiv \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}_a$$

$$\equiv \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_a \otimes A_{11} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_a \otimes A_{12}$$

$$+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_a \otimes A_{21} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_a \otimes A_{22}.$$

For instance, the matrix $L_k(u)$ can be written

$$L_k(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_a \otimes \begin{pmatrix} \sinh(u+\frac{\gamma}{2}) & \\ & \sinh(u-\frac{\gamma}{2}) \end{pmatrix}_k$$

$$+ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_a \otimes \begin{pmatrix} & \\ \sinh(\gamma) & \end{pmatrix}_k$$

$$+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_a \otimes \begin{pmatrix} & \sinh(\gamma) \\ & \end{pmatrix}_k$$

$$+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_a \otimes \begin{pmatrix} \sinh(u-\frac{\gamma}{2}) & \\ & \sinh(u+\frac{\gamma}{2}) \end{pmatrix}_k$$

Note that in the definition of the transfer matrix, we 'trace away' the auxiliary space, so in a way it was never there all along.

Note the following facts (which one ought verify):

$$(1)\ L_{ak}(u)\Big|_{u=\frac{\gamma}{2}} = \sinh(\gamma) \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_k \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_k & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_k \end{pmatrix}_a$$

$$= \sinh(\gamma)\ P_{ak}.$$

Here, $P_{ab} \in End(V_a \otimes V_b)$ satisfies the property that if $M_{ab} \in End(V_a \otimes V_b)$ then $P_{ab} M_{ab} P_{ab} = M_{ba}$. This element, P_{ab}, is called a permutation matrix.