Exactly Solvable Models
Lecture 1

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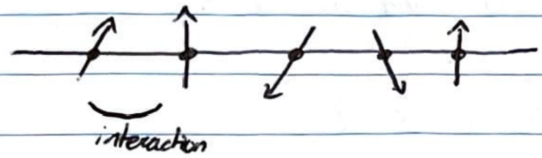
(1) Heisenberg Magnet (1928)

A simple model for magnetism



spin-$\frac{1}{2}$ particles

In any given spatial direction, the spin takes two possible values. We will denote them ± 1 (some people use $\pm\frac{1}{2}$, but this does not change the mathematics).

There are nearest neighbour interactions

§1.1. Pauli matrices

The Pauli matrices $\sigma^x, \sigma^y, \sigma^z$ are given by

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Collectively, they form a basis for the space of 2x2 complex traceless Hermitian (which means $\bar{A}^t = A$) matrices.
This space has a name: $\mathfrak{su}_2(\mathbb{C})$.

They satisfy the commutation relations:

$$[\sigma^x, \sigma^y] = 2i\sigma^z, \quad [\sigma^y, \sigma^z] = 2i\sigma^x, \quad [\sigma^z, \sigma^x] = 2i\sigma^y.$$

where the commutator is given by $[A, B] = AB - BA$.

§1.2 Spin states

The Pauli matrices have an action on the vector space

$$V = \mathrm{Span}_{\mathbb{C}}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

We shall denote $|+\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ and $|-\rangle = \begin{pmatrix}0\\1\end{pmatrix}$.

These are eigenvectors of σ^z:

$$\sigma^z|\pm\rangle = (\pm 1)|\pm\rangle$$

We will also define the dual space

$$V^* = \mathrm{Span}_{\mathbb{C}}\left\{\begin{pmatrix}1 & 0\end{pmatrix}, \begin{pmatrix}0 & 1\end{pmatrix}\right\}.$$

We shall denote $\langle +| = \begin{pmatrix}1 & 0\end{pmatrix}$ and $\langle -| = \begin{pmatrix}0 & 1\end{pmatrix}$.

We also use the standard inner product:

$$\langle \pm|\pm\rangle = 1 \quad \text{and} \quad \langle \pm|\mp\rangle = 0$$

§1.3 Copies of V

In what follows, we consider a system of N particles. We need a copy of V for each particle. We shall write:

$$\mathbb{V} = \underbrace{V \otimes \cdots \otimes V}_{N \text{ copies}} = V_1 \otimes \cdots \otimes V_n, \quad \text{where } V_i = V$$

This has basis:

$$\text{basis}(V) = \{ |i_1, \ldots i_n\rangle \mid i_k \in \{\pm\} \}$$

where $|i_1, \ldots, i_n\rangle = |i_1\rangle \otimes \cdots \otimes |i_n\rangle$.

The obvious extension to V^* also works.

We decorate Pauli matrices with subscripts to indicate in which space they act:

$$\sigma_k^{\omega} = \underbrace{\mathbb{1} \otimes \cdots \otimes \mathbb{1} \otimes \sigma^{\omega} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1}}_{\text{where } \sigma^{\omega} \text{ acts on the } k^{th} \text{ component}}, \qquad \omega \in \{x, y, z\},$$

and $\mathbb{1}$ is the identity map.

Exercise 1.3: Compute explicitly Basis(V) when $n=3$

§1.4 Hamiltonian

The Heisenberg spin chain/magnet is described by the Hamiltonian

$$H = \sum_{k=1}^{N} \left(J^x \sigma_k^x \sigma_{k+1}^x + J^y \sigma_k^y \sigma_{k+1}^y + J^z \sigma_k^z \sigma_{k+1}^z \right)$$

where $J^x, J^y, J^z \in \mathbb{R}$ and we define $\sigma_{N+1}^{\omega} = \sigma_1^{\omega}$.
The last condition is known as periodic boundary conditions.

One notes that this system is exactly solvable/integrable (whatever this means) for abitrary J^x, J^y, J^z. However, this is complicated. We restrict ourself to the variant where $J^x = J^y = 1$ but we leave J^z undetermined. This is called the XXZ

Heisenberg spin chain.

Exercise 1.4: Explicitly compute H for $N=3$.

§1.5 Goals

Our goals with respect to this model are the following:

- Compute the spectrum of H
  - ↳ The eigenvalues E of H
  - ↳ The eigenvectors $|\Psi\rangle$ of H.
- Compute correlation functions.
- Study the thermodynamic limit (i.e the limit as $N \to \infty$).

§1.6 Transfer matrix

The transfer matrix is going to operate on $\mathcal{V}$ with the following properties:

(1) It will commute with H
(In particular this will mean they share the same spectrum)

(2) It generates a set of N linearly independent commuting quantaties
(This is a Hallmark of integrability c.f Liouville-Arnold theorem).