

(1)(a) I wish to exhibit a bijection $S_n \times \binom{[n]}{k} \rightarrow [n] \times [n-1] \times \dots \times [n-k+1]$ where

$$\binom{[n]}{k} = \{\text{cardinality } k \text{ subsets of } [n]\}$$

$$[m] = \{1, \dots, m\}$$

For example, for $n=4$ and $k=3$.

$$\binom{[4]}{3} = \{\{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}$$

$$S_3 = \{(1\ 2\ 3), (1\ 3\ 2), (2\ 1\ 3), (2\ 3\ 1), (3\ 1\ 2), (3\ 2\ 1)\}$$

where we have written S_3 in one-line notation.

One should think of $\binom{[n]}{k}$ in fact as not subsets, but tuples with entries increasing. So, for instance

$$\{1, 2, 4\} = \{1, 4, 2\} = \{2, 1, 4\} = \dots$$

$$\updownarrow$$

$$(1, 2, 4)$$

When we pair this with an element of S_n , we should interpret this in the following way:

$$((3\ 1\ 2), (1, 3, 4)) \in S_n \times \binom{[n]}{k}$$

means I would like the elements 1, 3, 4, but choose 3 first, choose 4 second, and choose 1 third.

This will map to the element of $[n] \times [n-1] \times \dots \times [n-k+1]$ which describes the place which the number you would like lies at the time of choosing.

For example, the element $((3, 1, 2), (1, 3, 4))$ maps to

$$\begin{array}{cccc} 1 & 2 & \underline{3} & 4 \\ 1 & 2 & \underline{4} & \\ \underline{1} & 2 & & \end{array} \longleftrightarrow (3, 3, 1).$$

Here is a complete worked example for the $n=4$ and $k=3$ case. We have 24 elements to consider.

$(1, 2, 3) \quad \{1, 2, 3\}$ $\updownarrow$ $\begin{array}{cccc} \underline{1} & 2 & 3 & 4 \\ 2 & 3 & 4 & \\ \underline{3} & 4 & & \end{array}$ $\updownarrow$ $(1, 1, 1)$	$(1, 2, 3) \quad \{1, 2, 4\}$ $\updownarrow$ $\begin{array}{cccc} \underline{1} & 2 & 3 & 4 \\ 2 & 3 & 4 & \\ 3 & \underline{4} & & \end{array}$ $\updownarrow$ $(1, 1, 2)$
$(1, 2, 3) \quad \{1, 3, 4\}$ $\updownarrow$ $\begin{array}{cccc} \underline{1} & 2 & 3 & 4 \\ 2 & \underline{3} & 4 & \\ 1 & \underline{4} & & \end{array}$ $\updownarrow$ $(1, 2, 2)$	$(1, 2, 3) \quad \{2, 3, 4\}$ $\updownarrow$ $\begin{array}{cccc} 1 & \underline{2} & 3 & 4 \\ 1 & \underline{3} & 4 & \\ 1 & \underline{4} & & \end{array}$ $\updownarrow$ $(2, 2, 2)$

$(1\ 3\ 2)\ \{1,2,3\}$

$\updownarrow$
 $\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & \\ 2 & 4 & & \end{array}$
 $\updownarrow$

$(1,2,1)$

$(1\ 3\ 2)\ \{1,2,4\}$

$\updownarrow$
 $\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & \\ 2 & 3 & & \end{array}$
 $\updownarrow$

$(1,3,1)$

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$\updownarrow$
 $\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & \\ 2 & 3 & & \end{array}$
 $\updownarrow$

$(1,3,2)$

$(1\ 3\ 2)\ \{2,3,4\}$

$\updownarrow$
 $\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & \\ 1 & 3 & & \end{array}$
 $\updownarrow$

$(2,3,2)$

$(2\ 1\ 3)\ \{1,2,3\}$

$\updownarrow$
 $\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & \\ 3 & 4 & & \end{array}$
 $\updownarrow$

$(2,1,1)$

$(2\ 1\ 3)\ \{1,2,4\}$

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 $\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & \\ 3 & 4 & & \end{array}$
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$(2,1,2)$

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$(3,1,2)$

$(2\ 1\ 3)\ \{2,3,4\}$

$\updownarrow$
 $\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & \\ 1 & 4 & & \end{array}$
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$(3,2,2)$

$(2\ 3\ 1)\ \{1, 2, 3\}$

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2 4

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$(3, 1, 1)$

$(2\ 3\ 1)\ \{1, 2, 4\}$

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$(4, 1, 1)$

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$(4, 3, 2)$