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Advanced Discrete Mathematics
Problem Set 2

(1) Define $\binom{[n]}{k} = \{\text{subsets of } [n] \text{ of cardinality } k\}$.
$[n] = \{1, \dots, n\}$, $[0] = \emptyset$

Let $n \in \mathbb{Z}_{\geq 0}$ and $k \in \{0, \dots, n\}$.
Define

$$F : \binom{[n]}{k} \longrightarrow \binom{[n-1]}{k-1} \cup \binom{[n-1]}{k}$$

$$S \longmapsto S \setminus \{n\}.$$

One may show that this is a bijection with

$$F^{-1} : \binom{[n-1]}{k-1} \cup \binom{[n-1]}{k} \to \binom{[n]}{k}$$

$$S \in \binom{[n-1]}{k-1} \longmapsto S \cup \{n\}$$
$$S' \in \binom{[n-1]}{k} \longmapsto S'$$

Further $\binom{[n]}{0} = \{\text{subsets of } [n] \text{ with cardinality } 0\}$
$= \{\emptyset\}$

$\binom{[0]}{k} = \{\text{subsets of } [0] \text{ with cardinality } k\}$
$= \begin{cases} \{\emptyset\}, & k = 0 \\ \emptyset, & k \geq 1 \end{cases}$

$\binom{[n]}{n} = \{\text{subsets of } [n] \text{ with cardinality } n\}$
$= \{[n]\}$

$\binom{[n]}{m} = \emptyset$ when $m > n$.

From these pieces of information, if we can show $\binom{n}{k}$ satisfies the analogous conditions, we are done.

Notice that

$$\binom{n-1}{k} + \binom{n-1}{k-1}$$

$$= \frac{(n-1)!}{k!(n-k-1)!} + \frac{(n-1)!}{(k-1)!(n-k)!}$$

$$= \frac{(n-1)!(n-k)}{k!(n-k)!} + \frac{(n-1)!\,k}{k!(n-k)!}$$

$$= \frac{(n-1)!\,[n-k+k]}{k!(n-k)!}$$

$$= \frac{n!}{k!(n-k)!}$$

$$= \binom{n}{k}$$

and

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$$

$$\binom{0}{k} = \begin{cases} \frac{0!}{0!0!}, & k=0 \\ 0, & k \geq 1 \end{cases}$$

$$= \begin{cases} 1, & k=0 \\ 0, & k \geq 1 \end{cases}$$

$$\binom{n}{n} = \frac{n!}{n!0!} = 1$$

$$\binom{n}{m} = 0, \qquad m > n.$$

$\square$