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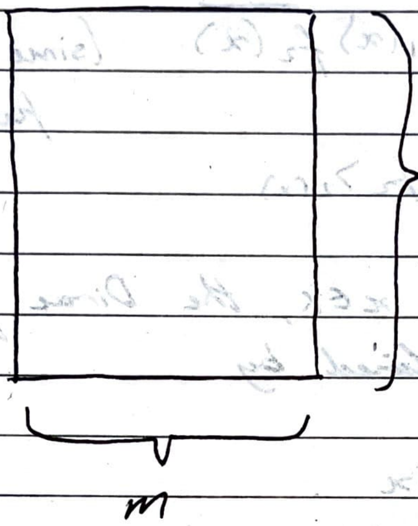
Advanced Discrete Maths Exercises 18th July 2025

(4) Fix $m,n \in \mathbb{Z}_{>0}$ with $m<n$ and let $(m^n) = \underbrace{(m, \dots, m)}_{n \text{ parts}}$.

(a) Determine the number of partitions μ such that $\mu \leq (m^n)$.

$(m^n) =$ 

One notices that given a multiset, say, $\{1,3,3,4,1,8\}$, there is only one such partition using those numbers, namely $(8\,4\,3\,3\,1\,1)$.

In this way, the number of partitions μ such that $\mu \leq (m^n)$ is the number of multisets of size n, with elements taken from the set $\{0,1,2,3,4,\dots,m\}$.

Famously, this is equivalent to the number of permutations of n ~~bars~~ stars and m bars.

For example, for (2^3) we are finding multisets of size 3 from elements taken from $\{0,1,2\}$ ($m=2$, $n=3$)

We have:

* * * | |	$\longleftrightarrow$	$\{0,0,0\}$	$\longleftrightarrow$	$(0\ 0\ 0)$
* * | * |	$\longleftrightarrow$	$\{0,0,1\}$	$\longleftrightarrow$	$(1\ 0\ 0)$
* * | | *	$\longleftrightarrow$	$\{0,0,2\}$	$\longleftrightarrow$	$(2\ 0\ 0)$
* | * * |	$\longleftrightarrow$	$\{0,1,1\}$	$\longleftrightarrow$	$(1\ 1\ 0)$
* | * | *	$\longleftrightarrow$	$\{0,1,2\}$	$\longleftrightarrow$	$(2\ 1\ 0)$
* | | * *	$\longleftrightarrow$	$\{0,2,2\}$	$\longleftrightarrow$	$(2\ 2\ 0)$
| * * * |	$\longleftrightarrow$	$\{1,1,1\}$	$\longleftrightarrow$	$(1\ 1\ 1)$
| * * | *	$\longleftrightarrow$	$\{1,1,2\}$	$\longleftrightarrow$	$(2\ 1\ 1)$
| * | * *	$\longleftrightarrow$	$\{1,2,2\}$	$\longleftrightarrow$	$(2\ 2\ 1)$
| | * * *	$\longleftrightarrow$	$\{2,2,2\}$	$\longleftrightarrow$	$(2\ 2\ 2)$

This number is $\binom{n+m}{n}$.

(b) Do some internet searches to get a feel for how this question is related to the Grassmannian of m-planes in n-space.

According to 'Affine approach to quantum schubert calculus' by Alexander Postnikov' the Grassmanian of m-planes in n-space has a cellular decomposition into Schubert cells. These cells are indexed by partitions λ whose Young diagrams fit inside the $(k\times(n-k))$-rectangle.

(5) For each $k \in \{1,2,3,4,5\}$ and each partition λ of size k, list the standard tableaux of shape λ. In each case, verify:

$$f_\lambda = \frac{n!}{\prod_{b \in \lambda} (\mathrm{arm}(b) + \mathrm{leg}(b) + 1)} \qquad \text{and} \qquad n! = \sum_{\lambda \in Y_n} f_\lambda^2 .$$

<u>$k=1$</u>

$\lambda_1 = \begin{array}{|c|}\hline \\ \hline\end{array}$ $\quad T_1^1 = \begin{array}{|c|}\hline 1 \\ \hline\end{array}$

<u>$k=2$</u>

$\lambda_1 = \begin{array}{|c|c|}\hline & \\ \hline\end{array}$ $\quad T_1^1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$

$\lambda_2 = \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}$ $\quad T_2^1 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline\end{array}$

<u>$k=3$</u>

$\lambda_1 = \begin{array}{|c|c|c|}\hline & & \\ \hline\end{array}$ $\quad T_1^1 = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$

$\lambda_2 = \begin{array}{|c|c|}\hline & \\ \hline \\ \cline{1-1}\end{array}$ $\quad T_2^1 = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ $\quad T_2^2 = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$

$\lambda_3 = \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}$ $\quad T_3^1 = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$

$\underline{k=4}$

$\lambda_1 = \ydiagram{4}$ $\quad T_1^1 = \begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau}$

$\lambda_2 = \ydiagram{3,1}$ $\quad T_2^1 = \begin{ytableau} 1 & 2 & 3 \\ 4 \end{ytableau}$ $\quad T_2^3 = \begin{ytableau} 1 & 3 & 4 \\ 2 \end{ytableau}$

$T_2^2 = \begin{ytableau} 1 & 2 & 4 \\ 3 \end{ytableau}$

$\lambda_3 = \ydiagram{2,2}$ $\quad T_3^1 = \begin{ytableau} 1 & 2 \\ 3 & 4 \end{ytableau}$

$T_3^2 = \begin{ytableau} 1 & 3 \\ 2 & 4 \end{ytableau}$

$\lambda_4 = \ydiagram{2,1,1}$ $\quad T_4^1 = \begin{ytableau} 1 & 2 \\ 3 \\ 4 \end{ytableau}$ $\quad T_4^3 = \begin{ytableau} 1 & 4 \\ 2 \\ 3 \end{ytableau}$

$T_4^2 = \begin{ytableau} 1 & 3 \\ 2 \\ 4 \end{ytableau}$

$\lambda_5 = \ydiagram{1,1,1,1}$ $\quad T_5^1 = \begin{ytableau} 1 \\ 2 \\ 3 \\ 4 \end{ytableau}$

For $k=1$: $\overset{LHS}{=} f_{\lambda_1} = 1$,

$$RHS = \frac{1!}{1} = 1$$

$LHS = 1! = 1$

$RHS = 1^2 = 1$

For $k=2$

(λ_1) : $LHS = 1$

$$RHS = \frac{2!}{2\cdot 1} = 1$$

$LHS = 2! = 2$

$RHS = 1^2 + 1^2 = 1+1 = 2$

(λ_2) : $LHS = 1$

$$RHS = \frac{2!}{2\cdot 1} = 1$$

For $k=3$

(λ_1) : $LHS = 1$

$$RHS = \frac{3!}{3\cdot 2\cdot 1} = 1$$

$LHS = 3! = 6$

$RHS = 1^2 + 2^2 + 1^2 = 1+4+1 = 6$

(λ_2) $LHS = 2$

$$RHS = \frac{3!}{3\cdot 1\cdot 1} = 2$$

(λ_3) $LHS = 1$

$$RHS = \frac{3!}{3\cdot 2\cdot 1} = 1$$

For $k=4$

(λ_1): LHS $= 1$

RHS $= \frac{4!}{4\cdot 3\cdot 2\cdot 1} = 1$

(λ_2): LHS $= \cancel{4}\ 3$

RHS $= \frac{4!}{4\cdot 2\cdot 1\cdot 1} = 3$

(λ_3): LHS $= 2$

RHS $= \frac{4!}{3\cdot 2\cdot 2\cdot 1} = 2$

(λ_4): LHS $= 3$

RHS $= \frac{4!}{4\cdot 1\cdot 2\cdot 1} = 3$

(λ_5): LHS $= 1$

RHS $= \frac{4!}{4\cdot 3\cdot 2\cdot 1} = 1$

LHS $= 4! = 4\cdot 3\cdot 2\cdot 1$
$= 24$

RHS $= 1^2 + 3^2 + 2^2 + 3^2 + 1^2$
$= 1 + 9 + 4 + 9 + 1$
$= 24$.