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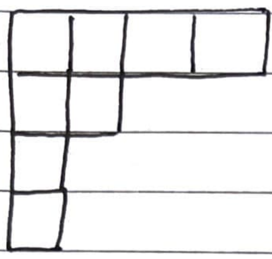
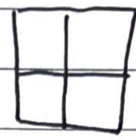
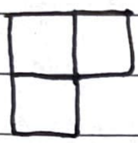
Advanced Discrete Mathematics
Exercise Sheet 1

① Give an example of two partitions λ and μ such that $\lambda \leq \mu$ and two partitions γ and β such that $\gamma \nleq \beta$.

 $= \mu = (4211)$

 $= \lambda = (22)$

 $= \beta = (21)$

 $= \gamma = (111)$

(2) For a partition λ, let $R(\lambda) = \sum_{b \in \lambda} (\text{coarm}_\lambda(b) - \text{coleg}_\lambda(b))$.

(Recall coarm is boxes to the left and
coleg is boxes above)

(a) Find partitions $\lambda \neq \mu$ such that $|\lambda| = |\mu|$ and $R(\lambda) = R(\mu)$

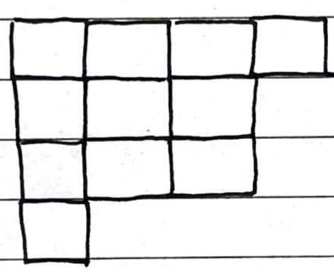
$|\lambda| = |\mu|$ means that λ and μ have the same number of boxes.

Placing coarm − coleg in each box, we get a grid that looks like:

0	1	2	3	4	5	6	7	. . .
-1	0	1	2	3	4	5	6	. . .
-2	-1	0	1	2	3	4	5	. . .
-3	-2	-1	0	1	2	3	4	. . .
-4	-3	-2	-1	0	1	2	3	. . .
-5	-4	-3	-2	-1	0	1	2	. . .
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	.

The diagonal is 0s, and this grid is anti-symmetric.
So, any symmetric partition $(\lambda = \lambda')$ will satisfy $R(\lambda) = 0$.

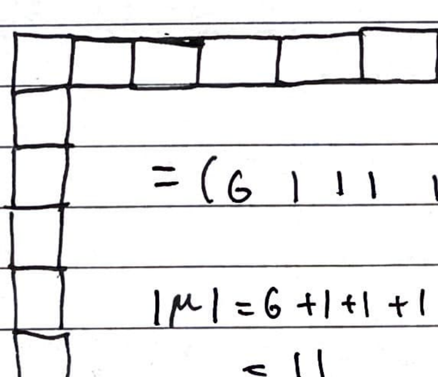
Take



$= (4\ 3\ 3\ 1) = \lambda.$

$|\lambda| = 4+3+3+1 = 11$ and $R(\lambda) = 0+1+2+3$
$+ -1 + 0 + 1$
$+ -2 + -1 + 0$
$+ -3$
$= 0$

also



$= (6\ 1\ 1\ 1\ 1\ 1) = \mu$

$|\mu| = 6+1+1+1+1+1$
$= 11$

$R(\mu) = 0+1+2+3+4+5$
-1
-2
-3
-4
-5
$= 0.$

(b) Do some internet searching to get a feel for how this question is related to Jucys-Murphy elements.

According to 'On complete functions in Jucys-Murphy Elements' by Valentin Féray, a Jucys-Murphy element J_i lives in $\mathbb{Z}[S_n]$ and is defined by

$$J_i = \sum_{j<i} (j\ i)$$ where $(j\ i)$ is the transposition in S_n swapping j and i.

They act diagonally on the Young basis of any irreducible representation V_λ, and the eigenvalue J_i on an element e_T of this basis (T is a standard tableau of shape λ) is simply given by the content (i.e coarm - coleg) of the box of T containing i.

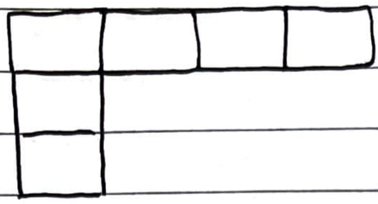
(3) For a partition λ, let

$$n(\lambda) = \sum_{b\in\lambda} (\mathrm{coleg}_\lambda(b) - 1)$$

(a) Find partitions $\lambda \neq \mu$ such that $|\lambda| = |\mu|$ and $n(\lambda) = n(\mu)$.

Placing coleg -1 in each box, we get a grid that looks like:

-1	-1	-1	-1	-1	-1	-1	-1	...
0	0	0	0	0	0	0	0	...
1	1	1	1	1	1	1	1	...
2	2	2	2	2	2	2	2	...
3	3	3	3	3	3	3	3	...
4	4	4	4	4	4	4	4	...
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

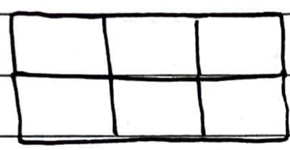
Take  $= \lambda = (4\ 1\ 1)$

$|\lambda| = 4+1+1 = 6$ and $n(\lambda) = -1\ -1\ -1\ -1$

$+0$

$+1$

$= -3$

and  $= \mu = (3\ 3)$

$|\mu| = 3+3 = 6$ and $n(\mu) = -1\ -1\ -1$

$+0\ +0\ +0$

$= -3$

(b) Do some internet searching to get a 'feel' for how this question is related to Springer fibers.

According to 'Richardson tableaux and components of springer fibers equal to Richardson varieties', if N is a nilpotent matrix of Jordan type λ which is in canonical form with block sizes weakly decreasing, the Springer fiber is

$$\mathcal{B}_\lambda = \{ (0 = F_0 \subseteq F_1 \subseteq \dots \subseteq F_n = \mathbb{C}^n) \in Fl_n(\mathbb{C}) \mid N(F_j) \subseteq F_{j-1} \text{ for all } j > 0 \}.$$

The irreducible components $\mathcal{B}_\sigma$ of $\mathcal{B}_\lambda$ are labelled by standard tableaux σ of shape λ, and every $\mathcal{B}_\sigma$ has dimension $n(\lambda) = \sum_{i=1}^{\ell}(i-1)\lambda_i$ (i.e. Σ coleg -1).