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Advanced Discrete Mathematics

Lecture 35

The gamma function

Integral formula: $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t}\, dt$.

Product formula: $\Gamma(\alpha) = \prod_{n=1}^{\infty} \frac{n}{n+\alpha} \left(\frac{n+1}{n}\right)^{\alpha}$

Difference equation: $\Gamma(\alpha+1) = \alpha \cdot \Gamma(\alpha)$

The q-gamma function

Integral formula: $\Gamma_q(a) = \int_0^{\frac{1}{1-q}} t^{a-1} E_q(-qt)\, d_q t$

Product formula: $\Gamma_q(a) = \prod_{n=1}^{\infty} \frac{[n]}{[n+a]} \left(\frac{[n+1]}{[n]}\right)^{a}$

$$= \frac{(1-q)^{1-a} (q;q)_\infty}{(q^a;q)_\infty}$$

Difference equation: $\Gamma_q(a+1) = [a]\, \Gamma_q(a)$

Remark: If $r \in \mathbb{Z}_{>0}$, then
$$\Gamma(r) = (r-1)!$$
$$\Gamma_q(r) = \frac{(q;q)_{r-1}}{(1-q)^{r-1}}.$$

The beta function

Integral formula: $\beta(x,y) = \int_0^1 t^{x-1}(1-t)^y \frac{dt}{(1-t)}$

Product formula: $\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ (analogue of binomial coefficients)

Difference equation: $\beta(x,y) = \beta(x,y+1) + \beta(x+1,y)$.

$$\beta(x+1,y) = \beta(x,y) \cdot \frac{x}{x+y}$$

$$\beta(x,y+1) = \beta(x,y) \cdot \frac{y}{x+y}$$

The q-beta function

Integral formula: $\beta_q(r,s) = \int_0^1 x^{r-1}(qx;q)_{s-1} \, d_q x$

Product formula: $\beta_q(r,s) = \frac{\Gamma_q(r)\,\Gamma_q(s)}{\Gamma_q(r+s)}$

Difference equation: $q^y \beta_q(x+1,y) + \beta_q(x,y+1) = \beta_q(x,y)$

$$\beta_q(x+1,y) = \beta_q(x,y) \cdot \frac{1-q^x}{1-q^{x+y}}$$

$$\beta_q(x,y+1) = \beta_q(x,y) \cdot \frac{1-q^y}{1-q^{x+y}}$$

The Gauss hypergeometric function

Power series $\quad {}_2F_1(a,b,c;z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{(a)_n (b)_n}{(c)_n} \cdot \frac{z^n}{n!}$

where $(a)_n = a(a+1)\cdots(a+n-1)$.

Integral formula $\quad {}_2F_1(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt$

Differential equation

$$z(1-z)\frac{d^2F}{dz^2} + (c-(a+b-1)z)\frac{dF}{dz} - abF = 0.$$

Then, ${}_2F_1(a,b,c;z)$ is a solution with $F(0)=1$.
General solutions are

$$F = C_1 \cdot {}_2F_1(a,b,c;z) + C_2 \cdot {}_2F_1(a+c-1, b-c+1, 2-c; z)$$

for $C_1, C_2 \in \mathbb{C}$.

The q-hypergeometric function

Power series $\quad {}_2\varphi_1(q^a, q^b, q^c; z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{(q^a;q)_n (q^b;q)_n}{(q^c;q)_n (q;q)_n} \cdot z^n$

Integral formula $\quad {}_2\varphi_1(a,b,c;z) \cdot \frac{\Gamma_q(c)}{\Gamma_q(b)\Gamma_q(c-b)} \int_0^1 t^{b-1} \frac{[(1-tz)^{-a}]}{[(1-tz)^{b-a}]} \frac{d_q t}{(1-t)}$

Difference equation $\quad (q^{a+b}z - q^{c-1})\varphi(q^2 z) + (-(q^a+q^b)z + q^{c-1}+1) + (z-1)\varphi(z) = 0$

Exponential function

$${}_0F_0(1, z) = \exp(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{1}{n!} z^n .$$

$$\exp(-z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{(-1)^n}{n!} z^n .$$

$$\frac{dg}{dz} = g$$

q-Exponential function

$${}_0\varphi_0(1; z, q) = E_q(z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{1}{(q;q)_n} \cdot z^n = (z;q)_\infty^{-1}$$

$$E_q(-z) = \sum_{n \in \mathbb{Z}_{\geq 0}} \frac{(-1)^n q^{\binom{n}{2}}}{(q;q)_n} z^n = (z;q)_\infty$$

$$E_q(-qz;q) = \frac{1}{1-z} E_q(-z) .$$