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Advanced Discrete Mathematics
Lecture 34

The binomial theorem

The binomial theorem is:

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{n-k} \quad \text{for} \quad n \in \mathbb{Z}_{\geq 0}$$

It has an application in showing the following property of the exponential function:

$$\text{if } xy = yx, \quad \text{then} \quad e^{x+y} = e^x e^y.$$

Both $(1+x)^n$ and e^x can be written as solutions to the following differential equations, respectively:

$$\frac{d(1+x)^n}{dx} = \frac{n}{1+x}(1+x)^n \quad \text{and} \quad \frac{de^x}{dx} = e^x.$$

More generally, we have

$$(1-z)^{-\alpha} = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{(\alpha)_k}{k!} z^k \quad \left(\text{Taylor series valid for } \alpha \in \mathbb{C}\right)$$

where $(\alpha)_k = \alpha(\alpha+1)\cdots(\alpha+k-1)$.

Let $(a;q)_k = (1-a)(1-aq)\cdots(1-aq^{k-1})$.
Then, define the q-analogue of the power function as:

$$[(1-z)^{-\alpha}] = \sum_{k \in \mathbb{Z}_{\geq 0}} \frac{(a;q)_k}{(q;q)_k} z^k. \quad \text{(notation from Etingof-Frenkel-Kirillov)}$$

Hypergeometric functions

${}_2F_1$ is the Gauss hypergeometric function.

The generalized hypergeometric function is

$${}_{r+1}F_r\left[\begin{matrix}\alpha_0,\dots,\alpha_r \\ \beta_1,\dots,\beta_r\end{matrix};z\right] = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(\alpha_0)_k\cdots(\alpha_r)_k}{(1)_k(\beta_1)_k\cdots(\beta_r)_k} \cdot z^k .$$

The q-hypergeometric function is

$${}_{r+1}\varphi_r\left[\begin{matrix}a_0,\dots,a_r \\ b_1,\dots,b_r\end{matrix};z\right] = \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(a_0;q)_k\cdots(a_r;q)_k}{(q;q)_k(b_1;q)_k\cdots(b_r;q)_k} \cdot z^k$$

(There is a generalization of Henze-Macdonal to symmetric matrices).

q-analogues of the binomial theorem

$(1+x)(1+xq)\cdots(1+xq^{n-1})$

$$= \sum_{k=0}^{n} x^k q^{\binom{k}{2}} \cdot \mathrm{Card}(\mathrm{Gr}(\mathbb{F}_q^n)_k)$$

$$= \sum_{k=0}^{n} x^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}$$

or

$$\prod_{i=1}^{n}(u+q^{i-1}z) = \sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} u^{n-k} z^k$$

$$= \sum_{k=0}^{n} q^{\binom{k}{2}} e_k(1,q,q^2,\dots,q^{n-1})\, u^{n-k} z^k .$$

Then,

$$\prod_{i=1}^{n} \frac{1}{u-q^{i-1}z} = \sum_{k\in\mathbb{Z}_{\geq 0}} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} u^{n-k} z^{k}$$

$$= \sum_{k\in\mathbb{Z}_{\geq 0}} h_k(1,q,q^2,\dots,q^{n-1})\, u^{n-k} z^{k}$$

and

$$\frac{(tz;q)_\infty}{(uz;q)_\infty} = \prod_{i=1}^{\infty} \frac{1-tzq^{i-1}}{1-uzq^{i-1}}$$

$$= \sum_{k\in\mathbb{Z}_{\geq 0}} \frac{(tu^{-1};q)_k}{(q;q)_k} u^{k} z^{k} .$$

Difference Equations

If $L(z;q,t,u) = \dfrac{(tz;q)_\infty}{(uz;q)_\infty}$, then

$$\boxed{L(z;q,t,u) = \frac{1-tz}{1-uz} L(qz;q,t,u).}$$