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Advanced Discrete Mathematics
Lecture 33

Hecke algebras and flag varieties

Define

$$\mathbb{G}(\mathbb{F}_q^n) = \{\mathbb{F}_q\text{-subspaces of } \mathbb{F}_q^n\}$$

partially ordered by inclusion.

Then,

$$\mathbb{G}(\mathbb{F}_q^n) = \bigsqcup_{k=0}^{n} \mathbb{G}(\mathbb{F}_q^n)_k, \quad \text{where} \quad \mathbb{G}(\mathbb{F}_q^n)_k = \left\{ \begin{array}{l} \mathbb{F}_q\text{-subspaces of } \mathbb{F}_q^n \\ \text{with dimension } k \end{array} \right\}$$

The Grassmannian of k-planes in $\mathbb{F}_q^n$ is $\mathbb{G}(\mathbb{F}_q^n)_k$.

The projective space of dimension $n-1$ is $\mathbb{P}^{n-1} = \mathbb{G}(\mathbb{F}_q^n)_1$.

The flag variety is

$$\begin{aligned} Fl(\mathbb{F}_q^n) &= \{\text{flags in } \mathbb{F}_q^n\} \\ &= \{\text{maximal chains in } \mathbb{G}(\mathbb{F}_q^n)\} \\ &= \{\emptyset \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n \mid \dim(V_i) = i\}. \end{aligned}$$

Let $G = GL_n(\mathbb{F}_q)$

$$B = \left\{ \begin{pmatrix} * & & * \\ & * & \\ & & * \end{pmatrix} \in GL_n(\mathbb{F}_q) \right\} \qquad P_k = \left\{ \begin{pmatrix} \underbrace{*}_{k} & * \\ & * \end{pmatrix} \in GL_n(\mathbb{F}_q) \right\}$$

Let $e_1,\dots,e_n \in \mathbb{F}_q^n$ be given by $e_i = (0,\dots,0,\overset{i^{th}}{1},0,\dots,0)$ and

$$E_k = \operatorname{span}\{e_1,\dots,e_k\}, \qquad k \in [1,n]$$

$$E = (\emptyset \subsetneq E_1 \subsetneq \cdots \subsetneq E_n) \in Fl(\mathbb{F}_q^n)$$

<u>Theorem</u> The maps

<u>(a)</u> $\begin{aligned} G/B &\to Fl(\mathbb{F}_q^n) \\ gB &\mapsto gE \end{aligned}$ <u>(b)</u> $\begin{aligned} G/P_k &\to \mathbb{G}(\mathbb{F}_q^n)_k \\ gP_k &\mapsto gE_k \end{aligned}$ <u>(c)</u> $\begin{aligned} G/P_1 &\to \mathbb{P}^{n-1} \\ gP_1 &\to gE_1 \end{aligned}$

are all well-defined bijections.

If $g = \begin{pmatrix} | & & | \\ g_1 & \cdots & g_n \\ | & & | \end{pmatrix} \in GL_n(\mathbb{F}_q)$, then

$$\begin{aligned} gE_k &= \operatorname{span}\{ge_1,\dots,ge_k\} \\ &= \operatorname{span}\{g_1,\dots,g_k\} \\ &:= \begin{bmatrix} | & & | \\ g_1 & \cdots & g_k \\ | & & | \end{bmatrix} \end{aligned}$$

and

$$gE_1 = \begin{bmatrix} | \\ g_1 \\ | \end{bmatrix} = [g_{11} : g_{21} : \cdots : g_{n1}]$$

Let $y_i(c) = 1 + (c-1)E_{ii} - E_{i+1,i+1} + E_{i,i+1} + E_{i+1,i}$ for $c \in \mathbb{C}^\times$ and $i \in [1, n-1]$.

Observe that

$$y_i(c) = \begin{array}{cc} \begin{array}{cccccc} & & i & i+1 & & \end{array} & \\ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & c & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} & \begin{array}{c} \\ \\ i \\ i+1 \\ \\ \\ \\ \end{array} \end{array} \qquad y_i(0) = s_i = \begin{array}{cc} \begin{array}{cccccc} & & i & i+1 & & \end{array} & \\ \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 0 & 1 & & \\ & & 1 & 0 & & \\ & & & & 1 & \\ & & & & & \ddots \\ & & & & & & 1 \end{pmatrix} & \begin{array}{c} \\ \\ i \\ i+1 \\ \\ \\ \\ \end{array} \end{array}$$

Theorem The symmetric group S_n is presented by generators

$s_1, \dots, s_{n-1}.$

with relations

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i \text{ if } i \notin \{j-1, j+1\}, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}.$$

One may check, by multiplying matrices, that the building relations hold:

(1) $$y_i(c_1)\, y_i(c_2) = \begin{cases} y_i(c_1 + c_2^{-1})\, h_i(c_1)\, h_{i+1}(-c_2^{-1})\, x_{i,i+1}(c_2^{-1}) \\ x_{i,i+1}(c_2) \end{cases}$$

where $$h_i(d) = \begin{array}{cc} \begin{array}{ccccc} & & i & & \end{array} & \\ \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & d & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix} & \begin{array}{c} \\ \\ i \\ \\ \\ \end{array} \end{array} = 1 + (d-1)E_{ii}$$

$$x_{ij}(c) = \begin{array}{cc} \begin{array}{cccc} & & j & \end{array} & \\ \begin{pmatrix} 1 & & c & \\ & \ddots & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} & \begin{array}{c} i \\ \\ \\ \\ \end{array} \end{array} = 1 + cE_{ij}$$

(2) $y_i(c_1)\, y_j(c_2) = y_j(c_2)\, y_i(c_1)$ if $i \notin \{j-1, j+1\}$

(3) $y_i(c_1)\, y_{i+1}(c_2)\, y_i(c_3) = y_{i+1}(c_3)\, y_i(c_1c_3 + c_2)\, y_{i+1}(c_1)$

Let $\mathbb{F}$ be a field

Theorem If $g \in GL_n(\mathbb{F})$, then there exists a unique $w \in S_n$ and unique $c_1, \dots, c_\ell \in \mathbb{C}$ and a unique $b \in B$

$$g = y_{i_1}(c_1) \cdots y_{i_\ell}(c_\ell)\, b$$

and $w = s_{i_1} \cdots s_{i_\ell}$ is the greedy reduced word for w.

Corollary

$$G = \bigsqcup_{w \in S_n} BwB \qquad \text{(Bruhat Decomposition)}$$

Greedy Reduced Word for w

$$w = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = s_2 \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = s_2 s_1 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$= s_2 s_1 s_3 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = s_2 s_1 s_3 s_2 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$= s_2 s_1 s_3 s_2 s_3 \cdot I.$$

Example

$$g = \begin{pmatrix} 7 & 6 & 2 & 4 \\ 1 & 8 & 7 & 9 \\ 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2\left(\tfrac{1}{8}\right) \begin{pmatrix} 7 & 6 & 2 & 4 \\ 8 & 6 & 3 & 5 \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & -\frac{5}{8} & -\frac{3}{8} \\ 0 & \frac{58}{8} & \frac{53}{8} & \frac{67}{8} \\ 0 & 1 & 1 & 2 \end{pmatrix} = y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & \frac{3}{4} & -\frac{5}{8} & -\frac{3}{8} \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right) y_2\left(\tfrac{3}{4}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{11}{8} & -\frac{15}{8} \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \end{pmatrix}$$

$$= y_2\left(\tfrac{1}{8}\right) y_1\left(\tfrac{7}{8}\right) y_3\left(\tfrac{58}{8}\right) y_2\left(\tfrac{3}{4}\right) y_3\left(\tfrac{11}{5}\right) \begin{pmatrix} 8 & 6 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -\frac{5}{8} & -\frac{49}{8} \\ 0 & 0 & 0 & \frac{58}{5} \end{pmatrix}$$