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16th Oct 2025

Advanced Discrete Mathematics
Lecture 32

Let $q, t \in \mathbb{C}$.
The double affine Hecke algebra $H_n(q,t)$ is the algebra generated by

$$T_0, T_1, \dots, T_{n-1}, T_\pi \quad \text{and} \quad Y_1, \dots Y_n$$

with relations

$$T_i^2 = (q-1)T_i + 1, \qquad T_i T_j = T_j T_i \text{ if } j \notin \{i-1, i+1\},$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \quad \text{(indices modulo } n\text{)}$$

$$Y_i Y_j = Y_j Y_i, \qquad Y_i = q T_{i-1}^{-1} Y_{i-1} T_{i-1}^{-1}$$

$$T_\pi T_i T_\pi^{-1} = T_{i+1} \qquad T_\pi Y_i T_\pi^{-1} = Y_{i+1}, \quad i \in \{1, \dots, n-1\}$$

$$T_\pi Y_n T_\pi^{-1} = t Y_1$$

Ask me a question?

Advice on writing referee reports.

It should stand alone: title, author, date, references.

Structure: (1) Summary of paper
(2) Context + positives + negatives
(3) Specific recommendations / improvements
(a) General things (put in X and Y)
(b) Specific things (Thm 2.3 has logical error)
(4) (Extra for Ass2) Improvements you implemented.

Theorem The following maps are bijections:

(a)
$$\left\{\begin{matrix}\text{partitions with}\\ n \text{ boxes}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{isom. classes of}\\ \text{simple } S_n\text{-modules}\end{matrix}\right\}$$
$$\lambda \longmapsto S_n^{\lambda}$$

(b)
$$\left\{\begin{matrix}\text{partitions with}\\ n \text{ boxes}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{isom. classes of}\\ \text{simple } H_n^{fin}(q)\text{-modules}\end{matrix}\right\}$$
$$\lambda \longmapsto H_n^{\lambda}$$

Define $q_r(x_1,\dots,x_n;u,t)$ by

$$\prod_{i=1}^{n} \frac{1-tx_iz}{1-ux_iz} = 1+(u-t)\sum_{r\in\mathbb{Z}_{\geq 1}} q_r(x_1,\dots,x_n;u,t)\,z^r$$

and p_r by

$$p_r = \prod_{i=1}^{n} x_i^r = q_r(x_1,\dots,x_n;u,u)\,u^{-(r-1)}$$

Define $\chi^{\lambda}_{u,t}(\mu)$ by

$$q_\mu = \sum_{\lambda} \chi^{\lambda}_{u,t}(\mu)\, s_\lambda$$

where $q_{(\mu_1,\dots,\mu_\ell)} = q_{\mu_1} \cdots q_{\mu_\ell}$ and $s_\lambda = \dfrac{a_{\lambda+\rho}}{a_\rho}$.

If $\mu = (\mu_1,\dots,\mu_\ell) \vdash n$, then define γ_μ by

$$\gamma_\mu = (s_1 s_2 \cdots s_{\mu_1 - 1})(s_{\mu_1+1} s_{\mu_1+2} \cdots s_{\mu_1+\mu_2-1}) \cdots (s_{\mu_1+\cdots+\mu_{\ell-1}+1} \cdots s_{n-1})$$

Similarly, define T_{γ_μ} by

$$T_{\gamma_\mu} = (T_1 T_2 \cdots T_{\mu_1-1})(T_{\mu_1+1} T_{\mu_1+2} \cdots T_{\mu_1+\mu_2-1}) \cdots (T_{\mu_1+\cdots+\mu_{\ell-1}+1} \cdots T_{n-1})$$

Theorem

(a) $tr(\gamma_\mu, S_n^\lambda) = \chi^{\lambda}_{1,1}(\mu)$

(b) $tr(T_{\gamma_\mu}, H_n^\lambda) = \chi^{\lambda}_{q,1}(\mu)$

Why is this true?

Let $V = \mathbb{C}^n$.
V has basis $\{e_1,\dots,e_n\}$ where $e_i = (0,\dots,0,\overset{i^{th}}{1},0,\dots,0)$.

Similarly $V^{\otimes k} = \underbrace{V \otimes \cdots \otimes V}_{k \text{ times}}$ has basis $\{e_{i_1} \otimes \cdots \otimes e_{i_k} \mid i_1, \dots, i_k \in [1,n]\}$.

Let $x_1, \dots, x_n \in \mathbb{C}^\times$.
Define D by

$$D : V^{\otimes k} \longrightarrow V^{\otimes k}$$

$$e_{i_1} \otimes \cdots \otimes e_{i_k} \longmapsto (x_{i_1} \cdots x_{i_k})(e_{i_1} \otimes \cdots \otimes e_{i_k})$$

Secretly: $x = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix} \in GL_n(\mathbb{C})$ and we think about

$GL_n(\mathbb{C})$ acting on $V^{\otimes k}$ by

$$g(e_{i_1} \otimes \cdots \otimes e_{i_k}) = (g e_{i_1}) \otimes \cdots \otimes (g e_{i_k}).$$

In this way, we get

$$\begin{aligned} D(e_{i_1} \otimes \cdots \otimes e_{i_k}) &= x(e_{i_1} \otimes \cdots \otimes e_{i_k}) \\ &= (x e_{i_1}) \otimes \cdots \otimes (x e_{i_k}) \\ &= (x_{i_1} \cdots x_{i_k})(e_{i_1} \otimes \cdots \otimes e_{i_k}) \end{aligned}$$

Define an action of S_k on $V^{\otimes k}$ by

$$w(e_{i_1} \otimes \cdots \otimes e_{i_k}) = e_{i_{w^{-1}(1)}} \otimes \cdots \otimes e_{i_{w^{-1}(k)}}$$

Notice that if $g \in GL_n(\mathbb{C})$ and $w \in S_k$, then $gw = wg$ as operators.

A small exercise

$\mathrm{tr}(D\gamma_r, V^{\otimes r})$

$$= \sum_{i_1,\dots,i_r \in [1,n]} D\gamma_r(e_{i_1} \otimes \cdots \otimes e_{i_r})\Big|_{\text{coefficient of } e_{i_1} \otimes \cdots \otimes e_{i_r}}$$

$$= \sum_{i_1,\dots,i_r \in [1,n]} D(e_{i_r} \otimes e_{i_1} \otimes \cdots \otimes e_{i_{r-1}})\Big|_{\text{coefficient of } e_{i_1} \otimes \cdots \otimes e_{i_r}}$$

$$= \sum_{\substack{i_1,\dots,i_r \in [1,n] \\ i_r = i_1 \\ i_1 = i_2 \\ \vdots \\ i_{r-1} = i_r}} D(e_{i_r} \otimes e_{i_1} \otimes \cdots \otimes e_{i_{r-1}})\Big|_{\text{coefficient of } e_{i_1} \otimes \cdots \otimes e_{i_r}}$$

$$= \sum_{i=1}^{n} D(e_i \otimes \cdots \otimes e_i)\Big|_{\text{coefficient of } e_i \otimes \cdots \otimes e_i}$$

$$= \sum_{i=1}^{n} x_i^r$$

$$= p_r .$$

This computation connects

symmetric functions

to

traces of actions of elements of S_n.

More generally,

$$\underline{tr(D\gamma_\mu, V^{\otimes k})} = p_{\mu_1} \cdots p_{\mu_\ell}$$

if $\mu = (\mu_1, \ldots, \mu_\ell) \vdash k$.

We will take the following fact for granted: As a $S_k \times GL_n(\mathbb{C})$ module,

$$\boxed{V^{\otimes k} \simeq \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} L(\lambda) \otimes S_k^\lambda}$$

where $L(\lambda)$ is the simple $GL_n(\mathbb{C})$-module associated to λ.

Taking traces of this decomposition,

$$LHS = tr(D\gamma_\mu, V^{\otimes k}) = p_\mu = \sum_\lambda \chi_{1,1}^\lambda(\mu)\, s_\lambda$$

$$RHS = tr\Big(D\gamma_\mu, \bigoplus_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} L(\lambda) \otimes S_k^\lambda\Big)$$

$$= \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} tr_{GL_n(\mathbb{C})}(D, L(\lambda)) \cdot tr_{S_k}(\gamma_\mu, S_k^\lambda)$$

$$= \sum_{\substack{\lambda \vdash k \\ \ell(\lambda) \leq n}} s_\lambda(x_1, \ldots, x_n)\, tr_{S_k}(\gamma_\mu, S_k^\lambda)$$

Equating coefficients (Frobenius Character Formula)

$$\chi_{1,1}^\lambda(\mu) = tr_{S_k}(\gamma_\mu, S_k^\lambda).$$