Gypsy Akhyar
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Advanced Discrete Mathematics
Lecture 31

The group algebra of the symmetric group $\mathbb{C}S_n$ is the algebra generated by

$$S_1, \dots, S_{n-1} \quad \text{and} \quad M_1, \dots, M_n$$

with relations

$$S_i^2 = 1, \quad S_i S_j = S_j S_i \text{ if } j \notin \{i-1, i+1\}, \quad S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$$

$$M_i M_j = M_j M_i, \quad M_1 = 0, \quad M_i = S_{i-1} M_{i-1} S_{i-1} + S_{i-1}$$

Let $q \in \mathbb{C}$.
The finite Hecke algebra $H_n^{fin}(q)$ is the algebra generated by

$$T_1, \dots, T_{n-1} \quad \text{and} \quad Y_1, \dots, Y_n$$

with relations

$$T_i^2 = (q - q^{-1}) T_i + 1, \quad T_i T_j = T_j T_i \text{ if } j \notin \{i-1, i+1\} \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}$$

$$Y_i Y_j = Y_j Y_i, \quad Y_1 = 1, \quad Y_i = T_{i-1}^{-1} Y_{i-1} T_{i-1}^{-1}$$

The affine Hecke algebra $H_n^{aff}(q)$ is the algebra generated by

$$T_1, \dots, T_{n-1} \quad \text{and} \quad Y_1, \dots, Y_n$$

with the same relations as $H_n^{fin}(q)$ EXCEPT you omit $Y_1 = 1$

So, we have a surjection:

$$\begin{array}{ccccc} H_n^{aff}(q) & \twoheadrightarrow & H_n^{fin}(q) & = & \dfrac{H_n^{aff}(q)}{\langle Y_1 = 1\rangle} \\ Y_1 & \longmapsto & 1 & & \\ Y_i & \longmapsto & y_i & & \\ T_i & \longmapsto & T_i & & \end{array}$$

<u>Another version of affine</u>

Let $G(\mathbb{F})$ be a group that depends on a field $\mathbb{F}$
(for example, take $G(\mathbb{F}) = GL_n(\mathbb{F})$)
Usually, you can then find a Lie algebra $g(\mathbb{F})$ for which we have

$$\exp : g(\mathbb{F}) \longrightarrow G(\mathbb{F}).$$

(for example, take $g = M_n(\mathbb{F})$).

Let $\mathbb{F}((t)) = \{ a_{-l} t^{-l} + a_{-l+1} t^{-l+1} + \dots \mid a_j \in \mathbb{F} \}$.

The <u>affine group</u> is $G(\mathbb{F}((t)))$

The <u>affine Lie algebra</u> is $g(\mathbb{F}((t)))$.

The <u>double affine Lie algebra</u> is $g(\mathbb{F}((q,t)))$.

<u>Modules</u>

<u>Theorem</u> The map

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{partitions with } n \\ \text{boxes} \end{array} \right\} & \longrightarrow & \left\{ \begin{array}{c} \text{isom. classes of simple} \\ S_n\text{-modules} \end{array} \right\} \\ \lambda & \longmapsto & S_n^{\lambda} \end{array}$$

is a bijection.

Here, S_n^λ has $\mathbb{C}$-basis $\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$ and

$$M_i v_T = ct(T^{-1}(i))\, v_T$$

and

$$\left(s_i + \frac{1}{M_i - M_{i+1}}\right) v_T = \left(1 + \frac{1}{M_i - M_{i+1}}\right) v_{s_i T}$$

where $s_i T$ is the same as T but with i and $i+1$ switched and $v_{s_i T} = 0$ if T is not standard.

Remarks (which make the proof easy)

(a) v_T is a simultaneous eigenvector for $M_1, \dots, M_n$.

(b) The operators $\left(s_i + \frac{1}{M_i - M_{i+1}}\right)$ "move v_T to $v_{s_i T}$".

Theorem The map

$$\left\{\begin{array}{c}\text{partitions with} \\ n \text{ boxes}\end{array}\right\} \longrightarrow \left\{\begin{array}{c}\text{isom. classes of simple} \\ H_n^{fin}(q)\text{-modules}\end{array}\right\}$$
$$\lambda \longmapsto H_n^\lambda$$

is a bijection.

Here, H_n^λ has $\mathbb{C}$-basis $\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$ and

$$Y_i v_T = q^{ct(T^{-1}(i))}\, v_T$$

and

$$\left(T_i + \frac{q-q^{-1}}{1-y_i y_{i+1}^{-1}}\right) v_T = \left(q^{-1} + \frac{q-q^{-1}}{1-y_i y_{i+1}^{-1}}\right) v_{s_i T}$$

(the "moving operators" are called "Cherednik intertwiners").