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Advanced Discrete Mathematics
Lecture 30

Representations of S_n and $H_n(q)$

Representation Theory

Let $\mathbb{C}$ be a field.
A $\mathbb{C}$-algebra is a $\mathbb{C}$-vector space A with a multiplication

$$\begin{aligned} A \times A &\longrightarrow A \\ (a_1, a_2) &\longmapsto a_1 a_2 \end{aligned}$$ which is linear in both components, associative, has identity and satisfies distributive laws.

(A ring is a $\mathbb{Z}$-module (i.e abelian group) with a...)

A representation of A is an A-module.

An A-module is a $\mathbb{C}$-vector space M with an A-action

$$\begin{aligned} A \times M &\to M \\ (a, m) &\mapsto am \end{aligned}$$ which is linear in both components, associative, has identity and satisfies distributive laws.

A simple, or irreducible, A-module is an A-module M which has no submodules except 0 and M.

First goal: Find simple A-modules.

Let G be a group.
A representation of G is a G-module.

A G-module is a $\mathbb{C}G$-module.

The <u>group algebra</u> of G is the $\mathbb{C}$-vector space

$$\mathbb{C}G = \mathbb{C}\text{-span}\{g \in G\}$$

(i.e. G is a basis of $\mathbb{C}G$)

with multiplication determined by the multiplication in G.

<u>Representations of $\mathbb{C}S_n$</u>

The <u>algebra $\mathbb{C}S_n$</u> is the algebra given by generators

$$s_1, \dots, s_{n-1} \quad \text{and} \quad M_1, \dots, M_n$$

with relations

$$s_i^2 = 1, \qquad s_i s_j = s_j s_i \text{ if } j \notin \{i-1, i+1\}, \qquad s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$$

$$M_i M_j = M_j M_i, \qquad M_1 = 0 \quad \text{and} \quad M_i = s_{i-1} M_{i-1} s_{i-1} + s_{i-1}$$

($M_1, \dots, M_n$ are 'Jucys-Murphy' elements.
Including the M_is is the Okounkov-Vershik method)

<u>Review</u> of our 'motivation' for partitions.

<u>Theorem</u> The map

$$\left\{ \begin{matrix} \text{partitions with} \\ n \text{ boxes} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{conjugacy classes in} \\ S_n \end{matrix} \right\}$$

$$\mu \longmapsto [\gamma_\mu]$$

where if $\mu = (\mu_1, \dots, \mu_l)$, then

$$\gamma_\mu = \gamma_{\mu_1} \cdots \gamma_{\mu_l} \qquad \text{and} \qquad \gamma_r = s_{r-1} \cdots s_1$$

is a bijection

Today's Theorem

Define $\widehat{S}_n^\lambda = \{\text{set of standard tableaux } T \text{ of shape } \lambda\}$.

Theorem The map

$$\left\{\begin{matrix}\text{partitions with} \\ n \text{ boxes}\end{matrix}\right\} \longrightarrow \left\{\begin{matrix}\text{isomorphism classes of} \\ \text{simple } \mathbb{C}S_n\text{-modules}\end{matrix}\right\}$$

$$\lambda \longmapsto [S_n^\lambda]$$

is a bijection

Here,

$$S_n^\lambda = \mathbb{C}\text{-span}\{ v_T \mid T \in \widehat{S}_n^\lambda \}$$

with $\mathbb{C}S_n$-action determined by

$$M_i v_T = ct(T^{-1}(i))\, v_T \qquad s_i \cdot v_T = (s_i)_{TT} \cdot v_T + (1 + (s_i)_{TT}) \cdot v_{s_i T}$$

and

$$(s_i)_{TT} = \frac{1}{ct(T^{-1}(i+1)) - ct(T^{-1}(i))}$$

and $s_i T$ is T except that i and $i+1$ are switched
and $v_{s_i T} = 0$ if $s_i T$ is not standard.

The Hecke case

Let $q \in \mathbb{C}^\times$.
The *Hecke algebra* $H_n(q)$ is the $\mathbb{C}$-algebra generated by

$$T_1, T_2, \dots, T_{n-1} \quad \text{and} \quad Y_1, Y_2, \dots, Y_n$$

with relations

$$T_i^2 = (q - q^{-1}) T_i + 1, \qquad T_i T_j = T_j T_i \quad \text{if } j \in \{i-1, i+1\}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \qquad Y_1 = 1, \qquad Y_i = T_{i-1}^{-1} Y_{i-1} T_i^{-1},$$

$$Y_i Y_j = Y_j Y_i.$$