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Advanced Discrete Mathematics
Lecture 29

↗ from problem solving session

Theorem The map

$$\{\text{partitions of } n\} \qquad \left\{\begin{array}{l}\text{isomorphism classes of} \\ \text{simple } S_n\text{-modules}\end{array}\right\}$$

$$\lambda \longmapsto S_n^\lambda$$

given by either

(1) $S_n^\lambda = \mathbb{C} S_n P_\lambda N_\lambda$, where

$$P_\lambda = \sum_{w \in \text{row stabilisers}} w \qquad N_\lambda = \sum_{v \in \text{column stabilisers}} \det(v) \cdot v$$

(2) $S_n^\lambda = \mathbb{C}\text{-span}\{v_T \mid T \text{ standard tableau of shape } \lambda\}$
with S_n-action given by

$$s_i \cdot v_T = \frac{1}{ct(T^{-1}(i+1)) - ct(T^{-1}(i))} v_T$$

$$+ \left(1 + \frac{1}{ct(T^{-1}(i+1)) - ct(T^{-1}(i))}\right) v_{s_i T}$$

where $ct(r,c) = c - r$ and $s_i T$ is T with i and $i+1$ switched and $v_{s_i T} = 0$ if $s_i T$ is not standard.

For $n, r \in \mathbb{Z}_{>0}$, define

$$\hat{q}_r(x_1, \dots, x_n; u, t) = \sum_{k=0}^{r-1} \sum_{i_r \le \dots \le i_{k+1} > \dots > i_1} u^{r-1-k} (-t)^k x_{i_r} \cdots x_{i_1}$$

The Pieri rule says that

$$\hat{q}_r s_\mu = \sum_{\substack{\lambda \\ \lambda/\mu \text{ border strip} \\ \text{with } r \text{ boxes}}} u^{\#\text{cols}(\lambda/\mu) - 1} (-t)^{\#\text{rows}(\lambda/\mu) - 1} s_\lambda$$

A skew shape λ/μ is a

$\begin{cases} \text{border strip} \\ \text{vertical strip} \\ \text{horizontal strip} \end{cases}$

if λ/μ has at most one box

$\begin{cases} \text{in each diagonal} \\ \text{in each row} \\ \text{in each column} \end{cases}$

Recall

(elementary)

$$e_r(x_1, \dots, x_n) = \sum_{i_r > \dots > i_1} x_{i_r} \cdots x_{i_1} = \hat{q}_r(x_1, \dots, x_n; 0, -1)$$

(homogeneous)

$$h_r(x_1, \dots, x_n) = \sum_{i_r \le \dots \le i_1} x_{i_r} \cdots x_{i_1} = \hat{q}_r(x_1, \dots, x_n; 1, 0)$$

(power sum)

$$p_r(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i^r = \hat{q}_r(x_1, \dots, x_n; u, u) \cdot u^{-(r-1)}$$

Example

$$\hat{q}_2(x_1,x_2,x_3;u,t) = u(x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3)$$
$$-t(x_1x_2+x_1x_3+x_2x_3)$$

So,

$$e_2(x_1,x_2,x_3) = \hat{q}_2(x_1,x_2,x_3;0,-1)$$
$$= x_1x_2+x_1x_3+x_2x_3$$

$$h_2(x_1,x_2,x_3) = \hat{q}_2(x_1,x_2,x_3;1,0)$$
$$= x_1^2+x_2^2+x_3^2+x_1x_2+x_1x_3+x_2x_3$$

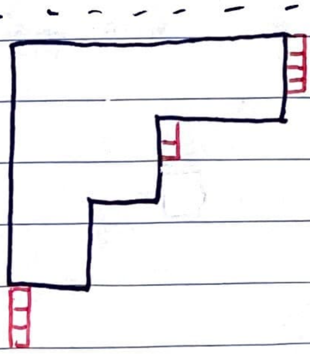
$$p_2(x_1,x_2,x_3) = \hat{q}_2(x_1,x_2,x_3;u,u)\,u^{-(2-1)}$$
$$= u(x_1^2+x_2^2+x_3^2)\cdot u^{-1}$$
$$= x_1^2+x_2^2+x_3^2$$

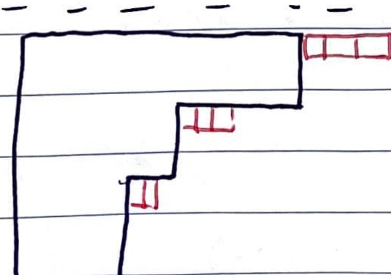
Specialisations

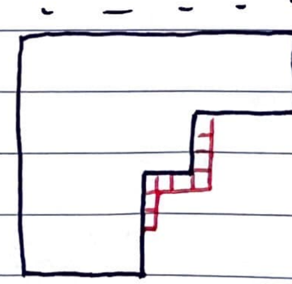
$$e_r s_\mu = \sum_{\lambda} s_\lambda$$
λ/μ vert strip w/ r boxes

$$h_r s_\mu = \sum_{\lambda} s_\lambda$$
λ/μ hori strip w/ r boxes

$$p_r s_\mu = \sum_{\lambda} s_\lambda$$
λ/μ connected border strip w/ r boxes







Example

Recall that u corresponds to # columns -1
$-t$ corresponds to # rows -1

So,

$$\hat{q}_1 = \hat{q}_1 \cdot s_\emptyset = s_{\ydiagram{1}}$$

$$\hat{q}_2 = \hat{q}_2 \cdot s_\emptyset = u\, s_{\ydiagram{2}} + (-t)\, s_{\ydiagram{1,1}}$$

$$\hat{q}_3 = \hat{q}_3 \cdot s_\emptyset = u^2 s_{\ydiagram{3}} + u(-t)\, s_{\ydiagram{2,1}} + (-t)^2 s_{\ydiagram{1,1,1}}$$

$$\hat{q}_4 = \hat{q}_4 \cdot s_\emptyset = u^3 s_{\ydiagram{4}} + u^2(-t)\, s_{\ydiagram{3,1}} + u(-t)^2 s_{\ydiagram{2,1,1}} + (-t)^3 s_{\ydiagram{1,1,1,1}}$$

Let $\mu = (\mu_1, \dots, \mu_l) \in \mathbb{Z}^l_{>0}$ with $\mu_1 \geq \dots \geq \mu_l > 0$ and $\mu_1 + \dots + \mu_l = k$.
(i.e μ is a partition with k boxes)

Define

$$\hat{q}_\mu = \hat{q}_{\mu_1} \hat{q}_{\mu_2} \cdots \hat{q}_{\mu_l}$$

Example

$$\hat{q}_{\ydiagram{2,2}} = \hat{q}_2 \cdot \hat{q}_2 = \hat{q}_2 \cdot \left(u\, s_{\ydiagram{2}} + (-t)\, s_{\ydiagram{1,1}}\right)$$

$$= u\, \hat{q}_2 \cdot s_{\ydiagram{2}} + (-t)\, \hat{q}_2 \cdot s_{\ydiagram{1,1}}$$

$$= u\left(u\, s_{\ydiagram{4}} + u(-t)\, s_{\ydiagram{3,1}} + (-t)\, s_{\ydiagram{2,1,1}} + u\, s_{\ydiagram{2,2}}\right)$$

$$-t\left(u\, s_{\ydiagram{3,1}} + u(-t)\, s_{\ydiagram{2,1,1}} + (-t)\, s_{\ydiagram{1,1,1,1}} + (-t)\, s_{\ydiagram{2,2}}\right)$$

Let λ and μ be partitions of k.
Then,

$$\boxed{\hat{q}_\mu = \sum_\lambda \chi^\lambda_{u,t}(\mu)\, s_\lambda}$$

where $\underline{\chi^\lambda_{u,t}(\mu)} = \sum_{Q \in SYT(\lambda)} wt^\mu(Q)$

and $\underline{wt^\mu(Q)} = \prod_{\substack{b \in \lambda \\ Q(b) \notin J(\mu)}} wt^\mu_Q(b)$

violation set

and $\underline{J(\mu)} = \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \dots + \mu_\ell\}$

and if $b \in \lambda$ with $Q(b) = j$, then

RSK!

$$\underline{wt^\mu_Q(b)} = \begin{cases} -t, & \text{if } j+1 \text{ is SW of } j \text{ in } Q \\ 0, & \text{if } j+1 \text{ is NE of } j \text{ in } Q \\ & \text{and } j+2 \text{ is SW of } j+1 \text{ in } Q \\ & \text{and } j+1 \notin J(\mu) \\ u, & \text{otherwise} \end{cases}$$

(Murnaghan - Nakayama rule on steroids)

Define a matrix $\underline{X_{u,t}} = \left(X^{\lambda}_{u,t}(\mu) \right)_{\lambda,\mu}$.

<u>For example</u>, if $k=2$,

$\lambda \backslash \mu$	$\ydiagram{2}$	$\ydiagram{1,1}$
$\ydiagram{2}$	u	1
$\ydiagram{1,1}$	$-t$	1

If $k=3$,

$\lambda \backslash \mu$	$\ydiagram{3}$	$\ydiagram{2,1}$	$\ydiagram{1,1,1}$
$\ydiagram{3}$	u^2	u	1
$\ydiagram{2,1}$	$0+u(-t)$	$u+(-t)$	$1+1$
$\ydiagram{1,1,1}$	$(-t)^2$	$(-t)$	1

The <u>character table of S_k</u> is $X_{1,1}$

The <u>character table of $H_k(q)$</u> is $X_{q,1}$ (Hecke algebra)

The <u>Kostka numbers</u> are $X^{\lambda}_{1,0}(\mu)$ (homogeneous to Schur)

<u>Contents of boxes</u>

The <u>content</u> of the box $b=(r,c)$ is $ct(b) = c-r$.

For normal people, these are the diagonal numbers of the box b.

A border strip is a skew shape λ/μ with at most one box in each diagonal.

A connected border strip is a border strip λ/μ such that

$$\mathrm{ct}(\lambda/\mu) \text{ is an interval in } \mathbb{Z}.$$