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Advanced Discrete Mathematics
Lecture 28

Review of generating symmetric functions

Goal: Pieri rules

Define $g_r = g_r(x) = g_r(x_1, \dots, x_n) = g_r(x_1, \dots, x_n; u, t, q)$ by

$$\prod_{i=1}^{n} \frac{(tx_i z; q)_\infty}{(ux_i z; q)_\infty} = \sum_{r \in \mathbb{Z}_{\geq 0}} g_r z^r$$

and $q_r(x_1, \dots, x_n; u, t)$ by

$$\prod_{i=1}^{n} \frac{1 - tx_i z}{1 - ux_i z} = \sum_{r \in \mathbb{Z}_{\geq 0}} q_r z^r.$$

Recall that since $(tx_i z; q)_\infty = (1 - tx_i z)(1 - tx_i zq)(1 - tx_i zq^2) \cdots$,
then we get $q_r(x_1, \dots, x_n; u, t) = g_r(x_1, \dots, x_n; u, t)$.

Possible paper: generalise today's lecture to g_r.

Pieri rule is a formula for

$$q_r s_\mu,$$

where s_μ is a Schur function.

Expand: $\displaystyle \prod_{i=1}^{n} \frac{1 - tx_i z}{1 - ux_i z} = \frac{1}{(1 - ux_1 z)} \cdots \frac{1}{(1 - ux_n z)} (1 - tx_n z) \cdots (1 - tx_1 z)$

to get

$$z^r q_r = (u-t) \sum_{i_r \le \cdots \le i_{k+1} > i_k > \cdots > i_1} u^{r-k} (-t)^k x_{i_r} \cdots x_{i_1} z^r$$

So,

factor from forcing i_{k+1} to be from u.

$= *(\le)$

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$$q_r = (u-t) \sum_{k=0}^{r-1} \sum_{i_r \le \cdots \le i_{k+1} > i_k > \cdots > i_1} u^{r-k} (-t)^k x_{i_r} \cdots x_{i_1}$$

Define

$$UD_r = \{ (i_r, \ldots, i_1) \mid i_1, \ldots, i_r \in \{1, \ldots, n\} \text{ and there exists } k \in \{0, \ldots, r-1\} \text{ with } i_r \le \cdots \le i_{k+1} > i_k > \cdots > i_1 \}$$

Let $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\ge 0}$ with $\mu_1 \ge \cdots \ge \mu_n$.

Define

$$B_n(\mu) = \{ \text{SSYT of shape } \mu \text{ filled from } \{1, \ldots, n\} \}.$$

Recall

$$s_\mu = \sum_{P \in B_n(\mu)} x^{wt(P)} = \sum_{P \in B_n(\mu)} x_1^{\#1\text{s in } P} \cdots x_n^{\#n\text{s in } P}.$$

is the Schur function.

Use RSK: $UD_r \times B_n(\mu) \longrightarrow \bigsqcup_{\lambda} B_n(\lambda)$.

λ border strip w/ r boxes

Example

Consider

becomes bottom row

$$(1,3,3,5,5,5,7,7,7,6,4,3,1) \in UD_{13}$$

and

$$\left(\begin{array}{llllll} 1 & 1 & 1 & 2 & 7 & 9 \\ 2 & 3 & 3 & 7 & & \\ 3 & 8 & 8 & & & \\ 7 & & & & & \end{array} \quad , \quad \begin{array}{llllll} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & 10 & & \\ 11 & 12 & 13 & & & \\ 14 & & & & & \end{array}\right) \in B_{14}(\mu).$$

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Inserting

$$\left(\begin{array}{cccccccc|ccccc} 27 & 26 & 25 & 24 & 23 & 22 & 21 & 20 & 19 & 18 & 17 & 16 & 15 \\ 1 & 3 & 3 & 5 & 5 & 5 & 7 & 7 & 7 & 6 & 4 & 3 & 1 \end{array}\right)$$

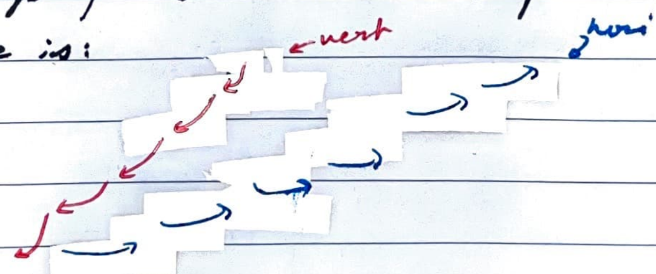
yields

$$\left(\begin{array}{lllllllllllll} 1 & 1 & 1 & 1 & 1 & 2 & 3 & 7 & 7 & 7 & 7 & 7 & 9 \\ 2 & 3 & 3 & 3 & 3 & 8 & & & & & & & \\ 3 & 5 & 5 & 6 & 8 & & & & & & & & \\ 4 & & & & & & & & & & & & \\ 5 & & & & & & & & & & & & \\ 7 & & & & & & & & & & & & \end{array} \quad , \quad \begin{array}{lllllllllllll} 1 & 2 & 3 & 4 & 5 & 6 & 15 & 22 & 23 & 24 & 25 & 26 & 27 \\ 7 & 8 & 9 & 10 & 16 & 21 & & & & & & & \\ 11 & 12 & 13 & 17 & 20 & & & & & & & & \\ 14 & & & & & & & & & & & & \\ 18 & & & & & & & & & & & & \\ 19 & & & & & & & & & & & & \end{array}\right)$$

The red always forms vertical strips

The blue always forms horizontal strips

General shape is:



A border strip is a skew shape λ/μ such that there is at most 1 box in each diagonal.

Theorem

$$q_r s_\mu = \sum_{\substack{\lambda/\mu \text{ border strip} \\ \text{with } r \text{ boxes}}} (u-t)\, u^{(\#\text{cols in } \lambda/\mu)-1} (-t)^{(\#\text{rows in } \lambda/\mu)-1} s_\lambda$$

elementary symmetric function

Since $\prod_{i=1}^{n} (1+x_i z) = \sum_{r\in\mathbb{Z}_{\geq 0}} e_r z^r$, then

$$e_r = \sum_{i_r > \dots > i_1} x_{i_r} \cdots x_{i_1} = q_r(x_1, \dots, x_n; 0, -1)$$

and so $e_r s_\mu = \sum_{\substack{\lambda/\mu \text{ vertical strip} \\ \text{with } r \text{ boxes}}}$.

A vertical strip is a skew shape λ/μ with at most 1 box in each row.