Gypsy Akhyar
25th Sep 2025

Advanced Discrete Mathematics (PS 26)

RSK going the other way

$$\left(\begin{array}{llllll} 1 & 1 & 1 & 2 & 7 & 9 \\ 2 & 3 & 3 & 7 & & \\ \boxed{3} & 8 & 8 & & & \\ 7 & & & & & \end{array} \;,\; \begin{array}{llllll} 1 & 2 & 2 & 4 & 6 & \boxed{6} \\ 3 & 3 & 4 & 5 & & \\ 4 & 6 & 6 & & & \\ 6 & & & & & \end{array}\right)$$

The right SSYT determines starting points.
We do the 6s, highest to lowest all the way to 1s highest to lowest right to left
The left SSYTs is where kicking occurs (kick $\leq$ smaller)

$$= \begin{pmatrix} 6 \\ 3 \end{pmatrix} \otimes \left(\begin{array}{lllll} 1 & 1 & 1 & 2 & 9 \\ 2 & 3 & 7 & 7 & \\ \boxed{3} & 8 & 8 & & \\ 7 & & & & \end{array} \;,\; \begin{array}{lllll} 1 & 2 & 2 & 4 & \boxed{6} \\ 3 & 3 & 4 & 5 & \\ 4 & 6 & 6 & & \\ 6 & & & & \end{array}\right)$$

$$= \begin{pmatrix} 6 & 6 \\ 3 & 3 \end{pmatrix} \otimes \left(\begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 7 & 7 & 9 \\ 3 & 8 & 8 & \\ \boxed{7} & & & \end{array} \;,\; \begin{array}{llll} 1 & 2 & 2 & 4 \\ 3 & 3 & 4 & 5 \\ 4 & 6 & \boxed{6} & \\ 6 & & & \end{array}\right)$$

$$= \begin{pmatrix} 6 & 6 & 6 \\ 3 & 3 & 7 \end{pmatrix} \otimes \left(\begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 7 & 7 & 9 \\ 3 & 8 & & \\ 8 & & & \end{array} \;,\; \begin{array}{llll} 1 & 2 & 2 & 4 \\ 3 & 3 & 4 & 5 \\ 4 & \boxed{6} & & \\ 6 & & & \end{array}\right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 \\ 3 & 3 & 7 & 8 \end{pmatrix} \otimes \left(\begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & 7 & 7 & 9 \\ 3 & & & \\ \boxed{8} & & & \end{array} \;,\; \begin{array}{llll} 1 & 2 & 2 & 4 \\ 3 & 3 & 4 & 5 \\ 4 & & & \\ \boxed{6} & & & \end{array}\right)$$

$$\begin{pmatrix} 6 & 6 & 6 & 6 & 6 \\ 3 & 3 & 7 & 8 & 8 \end{pmatrix} \otimes \left(\begin{array}{llll} 1 & 1 & 1 & 2 \\ 2 & \underline{7} & \underline{7} & \underline{9} \\ \boxed{3} & & & \end{array} , \begin{array}{llll} 1 & 2 & 2 & 4 \\ 3 & 3 & 4 & \boxed{5} \\ 4 & & & \end{array} \right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 \\ 3 & 3 & 7 & 8 & 8 & 3 \end{pmatrix} \otimes \left(\begin{array}{llll} \boxed{1} & \underline{1} & \underline{1} & \underline{2} \\ 2 & 7 & 9 & \\ 7 & & & \end{array} , \begin{array}{llll} 1 & 2 & 2 & \boxed{4} \\ 3 & 3 & 4 & \\ 4 & & & \end{array} \right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 \end{pmatrix} \otimes \left(\begin{array}{lll} 1 & 1 & 2 \\ 2 & \underline{7} & \underline{9} \\ \boxed{7} & & \end{array} , \begin{array}{lll} 1 & 2 & 2 \\ 3 & 3 & \boxed{4} \\ 4 & & \end{array} \right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 \end{pmatrix} \otimes \left(\begin{array}{lll} 1 & 1 & 2 \\ 2 & 9 & \\ \boxed{7} & & \end{array} , \begin{array}{lll} 1 & 2 & 2 \\ 3 & 3 & \\ \boxed{4} & & \end{array} \right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 4 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 \end{pmatrix} \otimes \left(\begin{array}{lll} 1 & 1 & 2 \\ \boxed{2} & \underline{9} & \end{array} , \begin{array}{lll} 1 & 2 & 2 \\ 3 & \boxed{3} & \end{array} \right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 4 & 3 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 & 2 \end{pmatrix} \otimes \left(\begin{array}{lll} 1 & 1 & 2 \\ \boxed{9} & & \end{array} , \begin{array}{lll} 1 & 2 & 2 \\ \boxed{3} & & \end{array} \right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 4 & 3 & 3 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 & 2 & 9 \end{pmatrix} \otimes \left(\boxed{1}\ \underline{1}\ \underline{2} ,\ 1\ 2\ \boxed{2} \right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 4 & 3 & 3 & 2 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 & 2 & 9 & 1 \end{pmatrix} \otimes \left(\boxed{1}\ \underline{2} ,\ 1\ \boxed{2} \right)$$

$$= \begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 4 & 3 & 3 & 2 & 2 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 & 2 & 9 & 1 & 1 \end{pmatrix} \otimes \left(\boxed{2} ,\ \boxed{1} \right)$$

$$=\begin{pmatrix} 6 & 6 & 6 & 6 & 6 & 5 & 4 & 4 & 4 & 3 & 3 & 2 & 2 & 1 \\ 3 & 3 & 7 & 8 & 8 & 3 & 1 & 7 & 7 & 2 & 9 & 1 & 1 & 2 \end{pmatrix} \quad \text{read } \begin{pmatrix} j \\ i \end{pmatrix}$$

This corresponds to the matrix $M \in M_{9\times 6}(\mathbb{Z}_{20})$ given by

$$M = \begin{pmatrix} 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

A forward example

Start with $M = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{pmatrix}$

This gives the array $\begin{pmatrix} 4 & 4 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 & 1 \\ 1 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 1 & 3 & 4 \end{pmatrix}$

Inserting gives:

$$\begin{pmatrix} 4 & 4 & 3 & 3 & 3 & 3 & 2 & 2 & 1 & 1 \\ 1 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 1 & 3 \end{pmatrix} \otimes (4, 1)$$

$$= \begin{pmatrix} 4 & 4 & 3 & 3 & 3 & 3 & 2 & 2 & 1 \\ 1 & 5 & 4 & 4 & 4 & 4 & 4 & 4 & 1 \end{pmatrix} \otimes (34, 11)$$

$$= \begin{pmatrix} 4 & 4 & 3 & 3 & 3 & 3 & 2 & 2 \\ 1 & 5 & 4 & 4 & 4 & 4 & 4 & 4 \end{pmatrix} \otimes (134, 111)$$

$$= \begin{pmatrix} 4 & 4 & 3 & 3 & 3 & 3 & 2 \\ 1 & 5 & 4 & 4 & 4 & 4 & 4 \end{pmatrix} \otimes \left(\begin{array}{lll} 1 & 3 & 4 \\ 4 & & \end{array} , \begin{array}{lll} 1 & 1 & 1 \\ 2 & & \end{array} \right)$$

$$= \begin{pmatrix} 4 & 4 & 3 & 3 & 3 & 3 \\ 1 & 5 & 4 & 4 & 4 & 4 \end{pmatrix} \otimes \left(\begin{array}{lll} 1 & 3 & 4 \\ 4 & 4 & \end{array} , \begin{array}{lll} 1 & 1 & 1 \\ 2 & 2 & \end{array} \right)$$

$$= \begin{pmatrix} 4 & 4 & 3 & 3 & 3 \\ 1 & 5 & 4 & 4 & 4 \end{pmatrix} \otimes \left(\begin{array}{llll} 1 & 3 & 4 & 4 \\ 4 & 4 & & \end{array} , \begin{array}{llll} 1 & 1 & 1 & 3 \\ 2 & 2 & & \end{array} \right)$$

$$= \begin{pmatrix} 4 & 4 & 3 & 3 \\ 1 & 5 & 4 & 4 \end{pmatrix} \otimes \left(\begin{array}{lllll} 1 & 3 & 4 & 4 & 4 \\ 4 & 4 & & & \end{array} , \begin{array}{lllll} 1 & 1 & 1 & 3 & 3 \\ 2 & 2 & & & \end{array} \right)$$

$$= \begin{pmatrix} 4 & 4 \\ 1 & 5 \end{pmatrix} \otimes \left(\begin{array}{lllllll} 1 & 3 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & & & & & \end{array} , \begin{array}{lllllll} 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ 2 & 2 & & & & & \end{array} \right)$$

$$= \begin{pmatrix} 4 \\ 1 \end{pmatrix} \otimes \left(\begin{array}{lllllll} 1 & 3 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & & & & & \\ 5 & & & & & & \end{array} , \begin{array}{lllllll} 1 & 1 & 1 & 3 & 3 & 3 & 3 \\ 2 & 2 & & & & & \\ 4 & & & & & & \end{array} \right)$$

$$= \left(\begin{array}{llllllll} 1 & 1 & 3 & 4 & 4 & 4 & 4 & 4 \\ 4 & 4 & & & & & & \\ 5 & & & & & & & \end{array} , \begin{array}{llllllll} 1 & 1 & 1 & 3 & 3 & 3 & 3 & 4 \\ 2 & 2 & & & & & & \\ 4 & & & & & & & \end{array} \right)$$