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Advanced Discrete Mathematics
Lecture 26

RSK and Schur functions

The bijections (which are weight preserving)

$$M_{m\times n}(\mathbb{Z}_{\geq 0}) \longleftrightarrow \left\{ \begin{array}{l} \text{sequences of pairs } \binom{i}{j} \text{ in} \\ \text{weakly decreasing order} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{pairs } (P,Q) \\ P \text{ filled from } \{1,\dots,n\} \\ Q \text{ filled from } \{1,\dots,m\} \\ \text{shape}(P) = \text{shape}(Q) \end{array} \right\}$$

give the Cauchy identities

$$\prod_{j=1}^{m}\prod_{i=1}^{n} \frac{1}{1-x_iy_j} = \prod_{j=1}^{m}\prod_{i=1}^{n}\left(1 + x_iy_j + (x_iy_j)^2 + \cdots\right)$$

$$= \sum_{\lambda} \sum_{\substack{P\in B_n(\lambda) \\ Q\in B_m(\lambda)}} x^{wt(P)} y^{wt(Q)}$$

$$= \sum_{\lambda} s_\lambda(x_1,\dots,x_n)\, s_\lambda(y_1,\dots,y_m)$$

We will now consider certain restrictions of this bijection.

First, restrict to matrices in $M_{m\times n}(\{0,1\})$ with one 1 in each row.

For example: $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \binom{3}{3} \otimes \binom{2}{1} \otimes \binom{1}{3}$

$$= \begin{pmatrix}3\\3\end{pmatrix} \otimes \begin{pmatrix}2\\1\end{pmatrix} \otimes \left(3 \;,\; 1 \right)$$

$$= \begin{pmatrix}3\\3\end{pmatrix} \otimes \left(1\;3 \;,\; 1\;2 \right)$$

$$= \left(\begin{matrix}1 & 3\\ 3 & \end{matrix} \;,\; \begin{matrix}1 & 2\\ 3 & \end{matrix} \right).$$

The condition of having one 1 in each row translates to two line arrays of the form

$$\begin{pmatrix} m & m-1 & \cdots & \cdots & 2 & 1 \\ i_m & i_{m-1} & \cdots & \cdots & i_2 & i_1 \end{pmatrix}$$

Recall that the top row of a two line array corresponds to the Q tableaux. This means that Q must be in fact a SYT.

We obtain:

$$\left\{ \begin{matrix} \text{matrices in } M_{m\times n}(\{0,1\}) \text{ with} \\ \text{one 1 in each row} \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} \text{pairs } (P,Q),\ \text{shape}(P) = \text{shape}(Q) \\ P \text{ SSYT from } \{1,\dots,n\},\ Q \text{ SYT from } \{1,\dots,m\} \end{matrix} \right.$$

This gives

pick where 1 is in each row

$$(x_1 + \cdots + x_n)^m y_1 \cdots y_m = \sum_{\substack{P \in B_n(\lambda) \\ Q \in B_m(\lambda) \\ |\lambda| = m \\ Q \text{ standard}}} x^{\mathrm{wt}(P)} y^{\mathrm{wt}(Q)}$$

$$= \sum_{\substack{P \in B_n(\lambda) \\ |\lambda| = m}} x^{\mathrm{wt}(P)} (y_1 \cdots y_m) f_\lambda$$

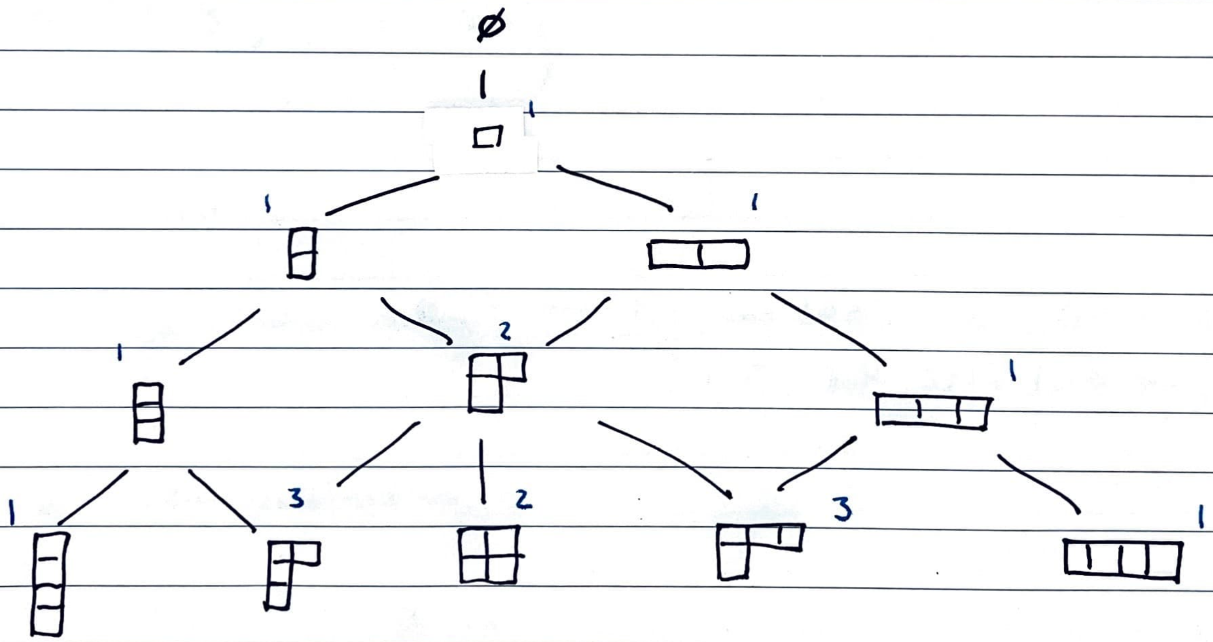
So,

$$(x_1 + \cdots + x_n)^m = \sum_{\substack{P \in B_n(\lambda) \\ |\lambda| = m}} x^{wt(P)} f_\lambda$$

$$= \sum_{\substack{\lambda \\ |\lambda| = m}} s_\lambda(x_1, \dots, x_n) f_\lambda .$$

where f_λ is the number of standard tableaux of shape λ.

Example Choose $m = 4$. So, we look at level 4 of the Young lattice.



So, $(x_1 + \cdots + x_n)^4 = s_{(1,1,1,1)} + 3s_{(2,1,1)} + 2s_{(2,2)} + 3s_{(3,1)} + s_{(4)}$.

Next, we restrict to permutation matrices.
These have one 1 in each row and column.

For *example*:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \otimes \begin{pmatrix} 2 \\ 1 \end{pmatrix} \otimes \left(3 \, , \; 1 \right)$$

$$= \begin{pmatrix} 3 \\ 2 \end{pmatrix} \otimes \left(1 \; 3 \, , \; 1 \; 2 \right)$$

$$= \left(\begin{array}{ll} 1 & 3 \\ 2 & \end{array} , \quad \begin{array}{ll} 1 & 2 \\ 3 & \end{array} \right)$$

We obtain

$$\{\text{permutation matrices}\} \longleftrightarrow \left\{ \begin{array}{l} \text{pairs } (P,Q), \ \text{shape}(P) = \text{shape}(Q) \\ P, Q \ \text{ both SYTs (with some filling)} \end{array} \right\}$$

So, taking cardinalities:

$$n! = \sum_\lambda f_\lambda^2$$

Example: $1^2 + 3^2 + 2^2 + 2^2 + 3^2 + 1^2 = 4!$