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Advanced Discrete Mathematics
Lecture 25

RSK and Cauchy Identity

If $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}_{\geq 0}^n$ with $\lambda_1 \geq \dots \geq \lambda_n$, then define

$$B_n(\lambda) = \left\{ \begin{array}{l} \text{SSYTs of shape } \lambda \\ \text{filled from } \{1, \dots, n\} \end{array} \right\}$$

Example:

$$B_3\left(\begin{array}{|c|c|}\hline \, & \, \\ \hline \, & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\right) = B_3(2,1,0)$$

$$= \left\{ \begin{array}{cccc} \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 1 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \\ \\ \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} & \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \end{array} \right\}$$

The *Length of λ* is $\ell(\lambda) = \#\{ i \mid \lambda_i \neq 0 \}$.

Let $m, n \in \mathbb{Z}_{>0}$ and suppose λ is a partition with $\ell(\lambda) \leq n$ and $\ell(\lambda) \leq m$.

Then,
$$s_\lambda(x) = s_\lambda(x_1, \dots, x_n) = \sum_{p \in B_n(\lambda)} x^{wt(p)}$$
$$= \sum_{p \in B_n(\lambda)} x_1^{\#1\text{s in } p} \cdots x_n^{\#n\text{s in } p}$$

and

$$s_\lambda(y) = s_\lambda(y_1,\dots,y_m) = \sum_{Q\in B_m(\lambda)} y^{wt(Q)}$$

$$= \sum_{Q\in B_m(\lambda)} y_1^{\#1\text{s in } Q} \cdots y_m^{\#m\text{s in } Q}.$$

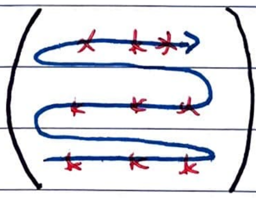
Now, before we do something with s_λ, consider the following bijection:

$$M_{m\times n}(\mathbb{Z}_{\geq 0}) \longleftrightarrow \left\{ \begin{matrix} \text{sequences of pairs } \binom{i}{j} \\ \text{in weakly decreasing order} \end{matrix} \right\}$$

$$A=(a_{ij}) \longmapsto \underbrace{\underbrace{\binom{m}{n}\otimes\cdots\otimes\binom{m}{n}}_{a_{mn}\text{ times}} \otimes \underbrace{\binom{m}{n-1}\otimes\cdots\otimes\binom{m}{n-1}}_{a_{mn-1}} \otimes\cdots}_{\text{reordered in weakly decreasing order}}$$

The <u>order</u> is $\begin{cases} \binom{i}{k} > \binom{j}{k}, & \text{if } i>j \\ \binom{i}{k} > \binom{i}{l}, & \text{if } k<l \end{cases}$

<u>Example:</u>

$\begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 4 \\ 0 & 3 & 0 \end{pmatrix} \in M_{3\times 3}(\mathbb{Z}_{\geq 0})$ has order 

The matrix maps to $\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{3}{2}\otimes\binom{2}{1}\otimes\binom{2}{2}\otimes\binom{2}{3}\otimes\binom{2}{3}\otimes\binom{2}{3}\otimes\binom{2}{3}$

$\otimes\binom{1}{1}\otimes\binom{1}{1}\otimes\binom{1}{1}\otimes\binom{1}{2}\otimes\binom{1}{2}.$

Weight of a matrix

Think of a matrix in the following way:

$$\begin{array}{c|ccccc} & x_1 & x_2 & x_3 & \cdots & x_n \\ \hline y_1 & & & & & \\ y_2 & & & & & \\ y_3 & & & & & \\ \vdots & & & & & \\ y_m & & & & & \end{array} \quad \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & \ddots & & & \\ a_{31} & & \ddots & & \\ \vdots & & & \ddots & \\ a_{m1} & & & & a_{mn} \end{pmatrix} = M \in M_{m\times n}(\mathbb{Z}_{\geq 0})$$

From here, define $\quad wt(M) = \prod_{i=1}^{n} \prod_{j=1}^{m} (x_i y_j)^{a_{ji}}.$

choose each entry

$$\text{Then, } \sum_{A \in M_{mn}(\mathbb{Z}_{\geq 0})} wt(A) = \prod_{i=1}^{n} \prod_{j=1}^{m} (1 + x_i y_j + (x_i y_j)^2 + \cdots)$$

$$= \prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1 - x_i y_j}.$$

generating function for $M_{m\times n}(\mathbb{Z}_{\geq 0})$

Example:

$$wt \begin{array}{c} \\ y_1 \\ y_2 \\ y_3 \end{array} \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{pmatrix} 3 & 2 & 0 \\ 1 & 1 & 4 \\ 0 & 3 & 0 \end{pmatrix} \end{array} = x_1^3 y_1^3 + x_2^2 y_2^2 + x_1 y_2 + x_2 y_2 + x_3^4 y_2^4 + x_2^3 y_3^3.$$

RSK insertion

We will 'insert' $\binom{i}{j}$ into a pair of column SSYTs.

A column SSYT looks like

$$\left.\begin{array}{|c|}\hline p_1 \\ \hline \vdots \\ \hline p_k \\ \hline \end{array}\right\} k \text{ boxes} \qquad \text{with } p_1 < \cdots < p_k$$

A pair of column SSYTs with the same number of boxes looks like

$$\left(\begin{array}{|c|}\hline p_1 \\ \hline \vdots \\ \hline p_k \\ \hline \end{array} \, , \, \begin{array}{|c|}\hline q_1 \\ \hline \vdots \\ \hline q_k \\ \hline \end{array} \right)$$

The following are the insertion rules:

$$\binom{i}{j} \otimes \left(\begin{array}{|c|}\hline p_1 \\ \hline \vdots \\ \hline p_k \\ \hline \end{array} \, , \, \begin{array}{|c|}\hline q_1 \\ \hline \vdots \\ \hline q_k \\ \hline \end{array} \right) = \left(\begin{array}{|c|}\hline p_1 \\ \hline \vdots \\ \hline j \\ \hline \vdots \\ \hline p_k \\ \hline \end{array} \, , \, \begin{array}{|c|}\hline q_1 \\ \hline \vdots \\ \hline q_k \\ \hline \end{array} \right) \otimes \binom{i}{p_l}$$

if $p_{l-1} < j \leq p_l$

$$\binom{i}{j} \otimes \left(\begin{array}{|c|}\hline p_1 \\ \hline \vdots \\ \hline p_k \\ \hline \end{array} \, , \, \begin{array}{|c|}\hline q_1 \\ \hline \vdots \\ \hline q_k \\ \hline \end{array} \right) = \left(\begin{array}{|c|}\hline p_1 \\ \hline \vdots \\ \hline p_k \\ \hline j \\ \hline \end{array} \, , \, \begin{array}{|c|}\hline q_1 \\ \hline \vdots \\ \hline q_k \\ \hline i \\ \hline \end{array} \right) \otimes (\emptyset, \emptyset)$$

if $j > p_k$

Example

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \begin{pmatrix} 2 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

This gives

$$\begin{pmatrix} 2 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 3 \end{pmatrix} \otimes (\emptyset, \emptyset)$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes (3, 1) \otimes (\emptyset, \emptyset)$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \otimes (1, 1) \otimes \begin{pmatrix} 1 \\ 3 \end{pmatrix} \otimes (\emptyset, \emptyset)$$

$$= \begin{pmatrix} 2 \\ 3 \end{pmatrix} \otimes (1, 1) \otimes (3, 1) \otimes (\emptyset, \emptyset)$$

$$= \left(\begin{matrix} 1 \\ 3 \end{matrix} , \begin{matrix} 1 \\ 2 \end{matrix} \right) \otimes (\emptyset, \emptyset) \otimes (3, 1) \otimes (\emptyset, \emptyset)$$

Removing the $(\emptyset, \emptyset)$, we get

$$\left(\begin{matrix} 1 \\ 3 \end{matrix} , \begin{matrix} 1 \\ 2 \end{matrix} \right) \otimes (3, 1) = \left(\begin{matrix} 1 & 3 \\ 3 & \end{matrix} , \begin{matrix} 1 & 1 \\ 2 & \end{matrix} \right)$$

from the two line array: $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 1 & 3 \end{pmatrix}$

Our process:

$$M_{m\times n}(\mathbb{Z}_{\geq 0}) \longrightarrow \left\{ \begin{array}{c} \text{two line arrays} \\ \text{in weakly decreasing} \\ \text{order} \end{array} \right\} \longrightarrow \left\{ \begin{array}{ll} \text{pairs } (P,Q), & P \text{ SSYT from } \{1,\dots,n\} \\ \text{shape}(P)=\text{shape}(Q) & Q \text{ SSYT from } \{1,\dots,m\} \end{array} \right\}$$

$$A \longmapsto \begin{pmatrix} i_1 \\ j_1 \end{pmatrix} \cdots \begin{pmatrix} i_L \\ j_L \end{pmatrix} \mapsto (P, Q)$$

The claim is this map is a bijection and is well-defined.

Theorem (Cauchy Identity)

$$\prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1}{1-x_i y_j} = \sum_{\lambda} S_\lambda(x) S_\lambda(y)$$

where λ satisfies $l(\lambda) \leq n$ and $l(\lambda) \leq m$.