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Advanced Discrete Mathematics
Lecture 24

Skew shapes and the Littlewood-Richardson rule

Let $n \in \mathbb{Z}_{>0}$.
A *lattice permutation* is a word

$$p = \boxed{i_1} \otimes \cdots \otimes \boxed{i_k} \qquad \in B(\square)^{\otimes k}$$

such that if $r \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n\}$, then

$$\#\left(\boxed{j} \text{ in } \boxed{i_1} \otimes \cdots \otimes \boxed{i_r}\right)$$

$$\geq \#\left(\boxed{j+1} \text{ in } \boxed{i_1} \otimes \cdots \otimes \boxed{i_r}\right)$$

where $B(\square) = \left(\boxed{1} \xrightarrow{1} \boxed{2} \longrightarrow \cdots \xrightarrow{n-1} \boxed{n}\right)$

Theorem A word p is a lattice permutation iff p is a highest weight element of $B(\square)^{\otimes k}$.

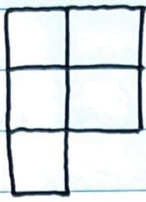
Skew shapes

Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n_{\geq 0}$ and $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$ be partitions. Assume $\mu \subseteq \lambda$ as boxes.

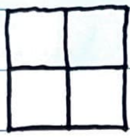
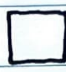
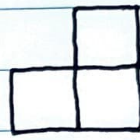
$$\lambda/\mu = \left\{ \begin{array}{l} \text{boxes in } \lambda \text{ that are} \\ \text{not in } \mu \end{array} \right\}$$

Examples

If $\lambda = (2\ 2\ 1)$ and $\mu = (1\ 1\ 0)$, then

$\lambda =$  and $\mu =$  and $\lambda/\mu =$ 

If $\lambda = (2\ 2\ 0)$ and $\mu = (1\ 0\ 0)$, then

$\lambda =$  and $\mu =$  and $\lambda/\mu =$ 

A SSYT of shape λ/μ filled from $\{1,\dots,n\}$ is a function $T: \lambda/\mu \to \{1,\dots,n\}$ such that

(a) If $(r,c), (r,c+1) \in \lambda/\mu$, then $T(r,c) \le T(r,c+1)$

(b) If $(r,c), (r+1,c) \in \lambda/\mu$, then $T(r,c) < T(r+1,c)$

Let $B(\lambda/\mu) = \{\text{SSYT of shape } \lambda/\mu \text{ filled from } \{1,\dots,n\}\}$

Example $n = 3$

$B\left((2\,2\,0)/(1\,0\,0) \right) = B\Big($  $\Big)$

$$= \left\{ \begin{array}{cccc} \begin{array}{|c|c|}\hline & 1 \\ \hline 1 & 2 \\ \hline\end{array} & \begin{array}{|c|c|}\hline & 1 \\ \hline 2 & 2 \\ \hline\end{array} & \begin{array}{|c|c|}\hline & 1 \\ \hline 2 & 3 \\ \hline\end{array} & \begin{array}{|c|c|}\hline & 1 \\ \hline 3 & 3 \\ \hline\end{array} \\ \begin{array}{|c|c|}\hline & 1 \\ \hline 1 & 3 \\ \hline\end{array} & \begin{array}{|c|c|}\hline & 2 \\ \hline 1 & 3 \\ \hline\end{array} & \begin{array}{|c|c|}\hline & 2 \\ \hline 2 & 3 \\ \hline\end{array} & \begin{array}{|c|c|}\hline & 2 \\ \hline 3 & 3 \\ \hline\end{array} \end{array} \right\}$$

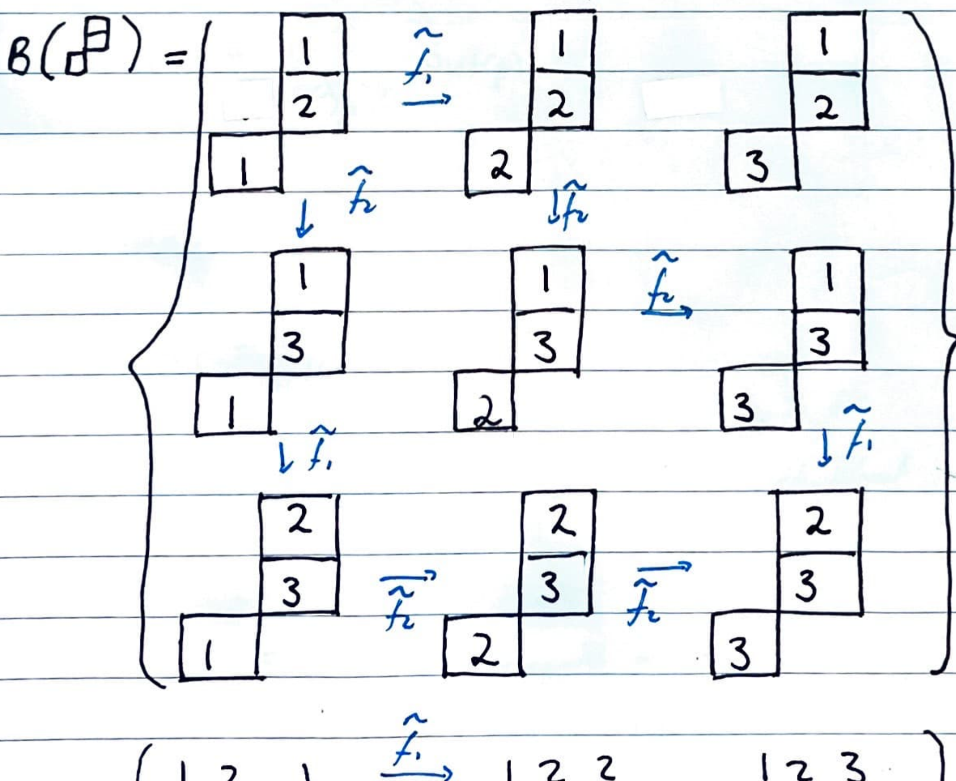
<u>Note</u>: $\lambda = \lambda/\emptyset$, and so we are generalising our previous ideas.

<u>Theorem</u> There exists a unique crystal structure (wt and $\tilde{f}_1, \dots, \tilde{f}_{n-1}$) such that

$$\begin{array}{ccc} B(\lambda/\mu) & \hookrightarrow & B(\square)^{\otimes k} \\ T & \longmapsto & \text{arabic reading of } T \end{array}$$

is a crystal morphism.

<u>For example</u>:



there are <u>two</u> highest weight vectors

$$\hookrightarrow \left\{ \begin{tikzcd} \underline{121} \arrow[r, "\tilde{f}_1"] \arrow[d, "\tilde{f}_2"] & 122 \arrow[d, "\tilde{f}_2"] & \underline{123} \\ 131 \arrow[d, "\tilde{f}_1"] & 132 \arrow[r, "\tilde{f}_2"] & 133 \arrow[d, "\tilde{f}_1"] \\ 231 \arrow[r, "\tilde{f}_2"] & 232 \arrow[r, "\tilde{f}_2"] & 233 \end{tikzcd} \right\} \subseteq B(\square)^{\otimes 3}$$

Remark: Before $B(\lambda)$ was always irreducible. This is not true for $B(\lambda/\mu)$.

A **Littlewood-Richardson filling** of λ/μ is a SSYT of shape λ/μ filled from $\{1, 2, \dots, n\}$ such that the arabic reading word of T is a lattice permutation (now we know this is the same as being a highest weight element of $B(\square)^{\otimes k}$; they didn't know this).

The **skew Schur function** $s_{\lambda/\mu}$ is

$$s_{\lambda/\mu} = \operatorname{char}(B(\lambda/\mu)).$$

Corollary (of yesterday)

$$s_{\lambda/\mu} = \sum_{p \in B(\lambda/\mu)^+} s_{\mathrm{wt}(p)}.$$

For **example**,

$$s_{(2,2,1)/(1,1)} = s_{(2,1)} + s_{(1,1,1)}.$$

For a partition $\nu = (\nu_1, \dots, \nu_n) \in \mathbb{Z}_{\geq 0}$ such that $\nu_1 \geq \dots \geq \nu_n$, define

$$\underline{B(\lambda/\mu)^+_\nu} = \left\{ \begin{array}{l} \text{highest weight elements of} \\ p \in B(\lambda/\mu) \text{ with } \mathrm{wt}(p) = \nu \end{array} \right\}$$

From here, we define the **Littlewood-Richardson coefficients**:

$$c^\lambda_{\mu\nu} = \operatorname{Card}(B(\lambda/\mu)^+_\nu).$$

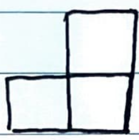
Then,

$$s_{\lambda/\mu} = \sum_{p \in B(\lambda/\mu)^+} s_{wt(p)} = \sum_{\nu} \sum_{p \in B(\lambda/\mu)^+_{\nu}} s_{wt(p)}$$

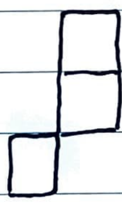
$$= \sum_{\nu} \sum_{p \in B(\lambda/\mu)^+_{\nu}} s_{\nu}$$

$$= \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu} .$$

<u>Example</u> If $\lambda = (2,2,0)$ and $\mu = (1,0,0)$, then

$\lambda/\mu =$  and $c^{\lambda}_{\mu\nu} = \begin{cases} 1, & \text{if } \nu = \text{} \\ 0, & \text{otherwise} \end{cases}$  .

If $\lambda = (2,2,1)$ and $\mu = (1,1,0)$, then

$\lambda/\mu =$  and $c^{\lambda}_{\mu\nu} = \begin{cases} 1, & \text{if } \nu = \text{} \\ 1, & \text{if } \nu = \text{} \\ 0, & \text{otherwise.} \end{cases}$