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Advanced Discrete Mathematics
PS23

Shuffle algebra

Let $V = \mathbb{C}\text{-span}\{e_1, \dots, e_n\}$.
The tensor algebra $T(V)$ is

$$T(V) = \text{span}\{ e_{i_1} \cdot \dots \cdot e_{i_k} \mid k \in \mathbb{Z}_{\geq 0},\ i_1, \dots, i_k \in \{1, \dots, n\}\}$$

with the operation of concatenation.

Define
$$\Delta : T(V) \to T(V) \otimes T(V)$$
$$e_i \mapsto e_i \otimes 1 + 1 \otimes e_i$$

with multiplication in $T(V) \otimes T(V)$ given by $(a \otimes b)(c \otimes d) = ac \otimes bd$.

Δ is an algebra homomorphism, which means that

$$\Delta(e_i e_j) = \Delta(e_i)\Delta(e_j), \quad \text{by definition,}$$

and

$$\begin{aligned}\Delta(e_i)\Delta(e_j) &= (e_i \otimes 1 + 1 \otimes e_i)(e_j \otimes 1 + 1 \otimes e_j) \\ &= (e_i \otimes 1)(e_j \otimes 1) + (e_i \otimes 1)(1 \otimes e_j) + (1 \otimes e_i)(e_j \otimes 1) \\ &\quad + (1 \otimes e_i)(1 \otimes e_j) \\ &= (e_i e_j \otimes 1) + (e_i \otimes e_j) + (e_j \otimes e_i) + 1 \otimes (e_i e_j).\end{aligned}$$

The shuffle algebra is the dual of $T(V)$. The multiplication in $T(V)^*$ is

$$T(V)^* \otimes T(V)^* \to T(V)^*, \quad (a, b) \longmapsto a \sqcup\!\sqcup\, b$$

This multiplication is best illustrated by example.
Define $x_1, \dots, x_n$ such that $x_i = e_i^*$.

Then $(x_1 x_7 x_4) ⧢ (x_2 x_2 x_9)$

$$= x_1 x_7 x_4 x_2 x_2 x_9 + x_1 x_7 x_2 x_4 x_2 x_9 + x_1 x_7 x_2 x_2 x_4 x_9$$
$$+ x_1 x_7 x_2 x_2 x_9 x_4 + x_1 x_2 x_7 x_2 x_4 x_9 + x_1 x_2 x_2 x_7 x_9 x_4$$
$$+ \text{(and so on...)}$$

The condition is that each pile ($x_1 x_7 x_4$ and $x_2 x_2 x_9$) must have their ordering preserved.

Schur polynomials

Let $G = GL_n(\mathbb{C})$
A G-module is a $\mathbb{C}$-vector space V with a G-action

$G \times V \to V$ such that if $g \in G$, then $\rho_g : V \to V$, $v \mapsto gv$ is a linear transformation
$(g, v) \mapsto gv$

Let V be a G-module. Then, G is irreducible, or simple, if V has no G-submodules except 0 and V.

Let M be a G-module (recall $G = GL_n$).
For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$, define

$$M_\mu = \{ m \in M \mid \operatorname{diag}(\mu_1, \dots, \mu_n) \cdot m = z_1^{\mu_1} \cdots z_n^{\mu_n} m \}$$ (weight spaces of M)

Then, there is a theorem that for rational G-modules:

$$M = \bigoplus_{\mu \in \mathbb{Z}^n} M_\mu \quad \text{and} \quad \operatorname{char}(M) = \sum_{\mu \in \mathbb{Z}^n} \dim(M_\mu)\, x^\mu .$$

Theorem Let M be an irreducible polynomial G-module.

(a) There exists a unique (up to scalar multiplication) $m^+ \in M$ such that $m^+ \neq 0$ and

$$\begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} m^+ = m^+ .$$ (highest weight vector)

(b) Let $\lambda = (\lambda_1, \dots, \lambda_n)$ be given by

$$\begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix} m^+ = z_1^{\lambda_1} \cdots z_n^{\lambda_n} m^+$$ (c.f. weight space decomposition)

Then, $\lambda_1 \geq \cdots \geq \lambda_n$ and $\mathrm{char}(M) = s_\lambda$.

Characters via traces:

Choose a basis of M which respects

$$M = \bigoplus_\mu M_\mu$$

i.e. $B_\mu = \{b_{\mu_1}, \dots, b_{\mu_{r_\mu}}\}$ is a $\mathbb{C}$-basis of M_μ and $B = \bigsqcup_\mu B_\mu$.

Let $g = \begin{pmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{pmatrix}$. Then, $\mathrm{tr}(\rho_g) = \sum_{b \in B} (\rho_g)_{bb}$.

$$= \sum_\mu \sum_{j=1}^{r_\mu} (\rho_g)_{b_{\mu_j} b_{\mu_j}}$$

$$= \sum_\mu \sum_{j=1}^{r_\mu} z_1^{\mu_1} \cdots z_n^{\mu_n} = \sum_\mu \dim(M_\mu) z^\mu = \mathrm{char}(M)\big|_{x_i = z_i} .$$