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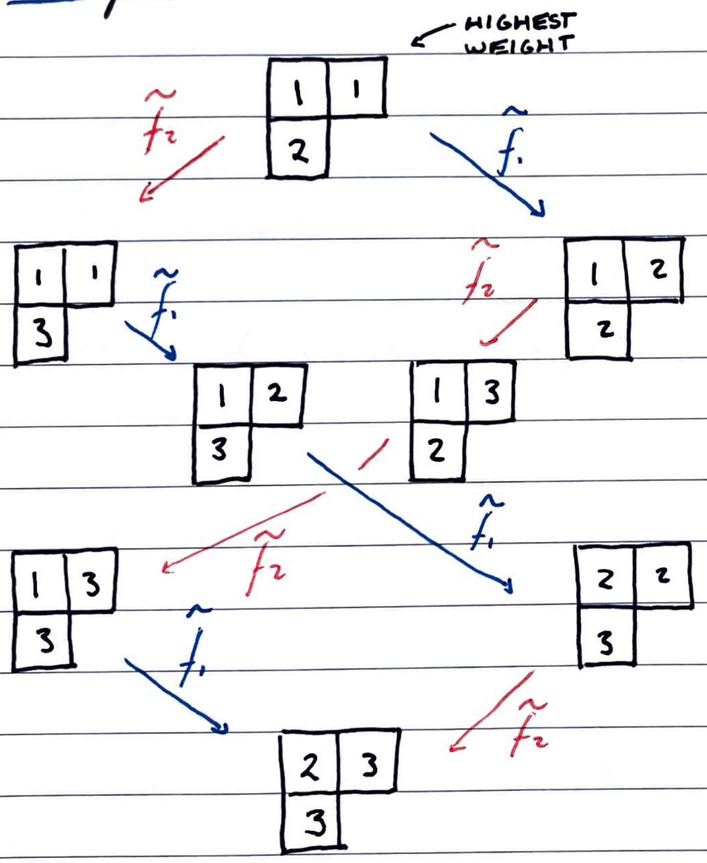
Advanced Discrete Mathematics
Lecture 23

Crystals and Schur functions

Recap: • Every crystal is a direct sum of components of $B(\square)^{\otimes k}$.

• The data of a crystal B includes $wt: B \to \mathbb{Z}^n$ and $\tilde{f}_1, \dots, \tilde{f}_{n-1}$ (which are matrices in $\{0,1\}$).

Example: $n=3$ and B is, with weights



$(2,1,0)$

$(2,0,1)$ $(1,2,0)$

$(1,1,1)$ $(1,1,1)$

$(1,0,2)$ $(0,2,1)$

$(0,1,2)$

So, $\mathrm{char}(B) = x_1^2 x_2 + x_1^2 x_3 + x_1 x_2^2 + x_1x_2x_3 + x_1x_2x_3 + x_1x_3^2 + x_2^2 x_3 + x_2x_3^2 = \sum_{p \in B} x^{wt(p)} = s_{(2,1,0)}$.

Define $\tilde{e}_1 = \tilde{f}_1^T, \ldots, \tilde{e}_{n-1} = \tilde{f}_{n-1}^T$.

A <u>highest weight element</u> of a crystal B is $p \in B$ such that $\tilde{e}_1 p = 0, \ldots, \tilde{e}_{n-1} p = 0$.

Let $\underline{B^+} = \{\text{highest weight elements in } B\}$.

<u>Theorem</u> Let B be a crystal.

<u>(a)</u> $\text{char}(B) \in \mathbb{C}[x_1, \ldots, x_n]^{S_n}$

<u>(b)</u> $\text{char}(B) = \sum_{p \in B^+} s_{wt(p)}$, where $s_{wt(p)}$ is the Schur function for $wt(p)$.

<u>Corollary</u> (Weyl's character formula)

Assume that B has a unique highest weight element p^+ and let $\lambda = wt(p^\lambda)$. Then,

$$\text{char}(B) = s_\lambda .$$

<u>Proof of (a)</u>

Let $j \in \{1, \ldots, n-1\}$ and $p \in B$.
Let

$$S_j(p) = \{ f_j^{d_j^-(p)} p, \ldots, \tilde{f}_j p, p, \tilde{e}_j p, \ldots, \tilde{e}_j^{d_j^+(p)} p \}$$

(i.e $S_j(p)$ is the j-string of p)

Define

$s_j p \in S_j(p)$ to be the unique element such that $wt(s_j p) = s_j wt(p)$.

(remains to show such an element exists).

Then, $s_j(s_j p) = p$.

So,

$$\begin{aligned} s_j(\operatorname{char}(B)) &= s_j\left(\sum_{p \in B} x^{wt(p)}\right) \\ &= \sum_{p \in B} x^{s_j wt(p)} \\ &= \sum_{p \in B} x^{wt(s_j p)} \\ &= \sum_{p \in B} x^{wt(p)} \qquad (s_j \text{ is bijective}) \\ &= \operatorname{char}(B). \end{aligned}$$

So, char(B) is a symmetric function.

Proof idea for (b)

Since char(B) is a symmetric function, then pushing it through the Boson-Fermion correspondence gives

$$\mathrm{char}(B)\, a_\rho = \mathrm{char}(B)\, e_- x^\rho$$

$$= e_- \mathrm{char}(B)\, x^\rho$$

$$= e_- \left(\sum_{p \in B} x^{wt(p)} x^\rho \right)$$

$$= e_- \left(\sum_{p \in B} x^{wt(p)+\rho} \right)$$

$$= \sum_{p \in B} a_{wt(p)+\rho} \, .$$

So, pushing it back,

$$a_\rho = \sum_{p \in B} \frac{a_{wt(p)+\rho}}{a_\rho}$$

$$= \sum_{p \in B} s_{wt(p)}$$

(argument by Littelmann)

$$= \sum_{p \in B'} s_{wt(p)} \, .$$

□

Going back to our example on page 1, we illustrate by example why $\sum_{p\in B} s_p = \sum_{p \in B^+} s_p$:

$$S_{(210)} = \frac{a_{(420)}}{a_\rho}$$

$$S_{(201)} = \frac{a_{(4,1,1)}}{a_\rho} \qquad S_{(120)} = \frac{a_{(330)}}{a_\rho}$$

$$S_{(111)} = \frac{a_{(321)}}{a_\rho} \qquad S_{(111)} = \frac{a_{(321)}}{a_\rho}$$

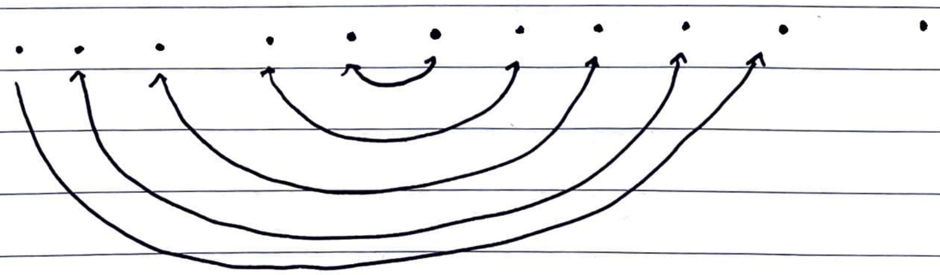
$$S_{(102)} = \frac{a_{(312)}}{a_\rho} \qquad S_{(021)} = \frac{a_{(231)}}{a_\rho}$$

$$S_{(012)} = \frac{a_{(222)}}{a_\rho}.$$

Yet, there is a lot of cancellation due to the fact that if $\gamma = \gamma_{i+1}$, then $a_\gamma = 0$.

Further, we have cancellation within i-string due to the property that $a_{s_j\gamma} = -a_\gamma$:

$$f_j^{d_j^-(p)} p \qquad \cdots \qquad e_j^{d_j^+(p)} p$$



For even strings, the central element will always give a 0 S_λ.